CK-12

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CHAPTER 10

CHAPTER 10 Perimeter and Area

Chapter Outline

Now that we have explored triangles, quadrilaterals, polygons, and circles, we are going to learn how to find the perimeter and area of each. First we will derive each formula and then apply them to different types of polygons and circles. In addition, we will explore the properties of similar polygons, their perimeters and their areas.

10.1 Triangles and Parallelograms

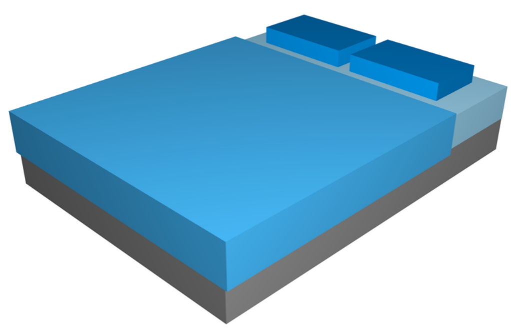
Learning Objectives

- Understand the basic concepts of area.
- Use formulas to find the area of triangles and parallelograms.

Review Queue

a. Define perimeter and area, in your own words.
b. Solve the equations below. Simplify any radicals.
    a. $x^2 = 121$
    b. $4x^2 = 80$
    c. $x^2 - 6x + 8 = 0$
c. If a rectangle has sides 4 and 7, what is the perimeter?

Know What? Ed's parents are getting him a new bed. He has decided that he would like a king bed. Upon further research, Ed discovered there are two types of king beds, an Eastern (or standard) King and a California King. The Eastern King has 76" $\times$ 80" dimensions, while the California King is 72" $\times$ 84" (both dimensions are *width* $\times$ *length*). Which bed has a larger area to lie on? Which one has a larger perimeter? If Ed is 6'4", which bed makes more sense for him to buy?



Areas and Perimeters of Squares and Rectangles

Perimeter: The distance around a shape. Or, the sum of all the edges of a two-dimensional figure.

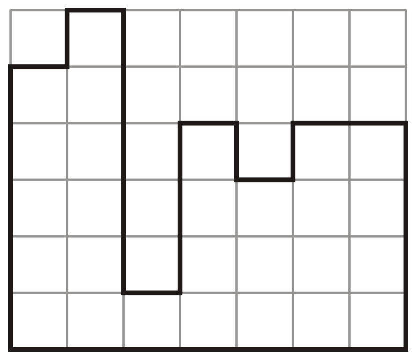
The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write "units."

Example 1: Find the perimeter of the figure to the left.

Solution: First, notice there are no units, but the figure is on a grid. Here, we can use the grid as our units. Count around the figure to find the perimeter. We will start at the bottom left-hand corner and go around the figure clockwise.

$$5+1+1+1+5+1+3+1+1+1+1+2+4+7$$

The answer is 34 units.



You are probably familiar with the area of squares and rectangles from a previous math class. Recall that you must always establish a unit of measure for area. Area is always measured in square units, square feet $(ft.^2)$, square inches $(in.^2)$. square centimeters $(cm.^2)$, etc. Make sure that the length and width are in the same units before applying any area formula. If no specific units are given, write "$units^2$."

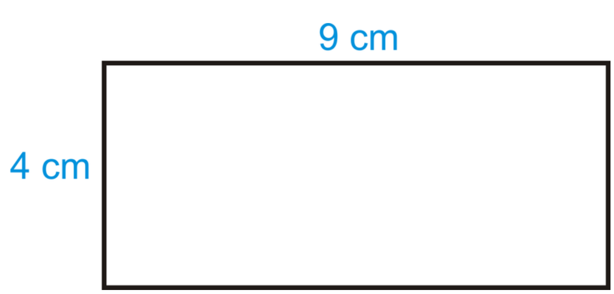
Example 2: Find the area of the figure from Example 1.

Solution: If the figure is not a standard shape, you can count the number of squares within the figure. If we start on the left and count each column, we would have:

$$5+6+1+4+3+4+4=27\ units^2$$

Area of a Rectangle: The area of a rectangle is the product of its base (width) and height (length) $A=bh$.

Example 3: Find the area and perimeter of a rectangle with sides 4 cm by 9 cm.



Solution: The perimeter is $4+9+4+9=36\ cm$. The area is $A=9\cdot 4=36\ cm^2$.

In this example we see that a formula can be generated for the perimeter of a rectangle.

Perimeter of a Rectangle: $P=2b+2h$, where b is the base (or width) and h is the height (or length).

If a rectangle is a square, with sides of length s, the formulas are as follows:

Perimeter of a Square: $P_{square} = 2s + 2s = 4s$

Area of a Square: $A_{sqaure} = s \cdot s = s^2$

Example 4: The area of a square is 75 in^2. Find the perimeter.

Solution: To find the perimeter, we need to find the length of the sides.

$$A = s^2 = 75\ in^2$$
$$s = \sqrt{75} = 5\sqrt{3}\ in$$
$$\text{From this, } P = 4\left(5\sqrt{3}\right) = 20\sqrt{3}\ in.$$

Area Postulates

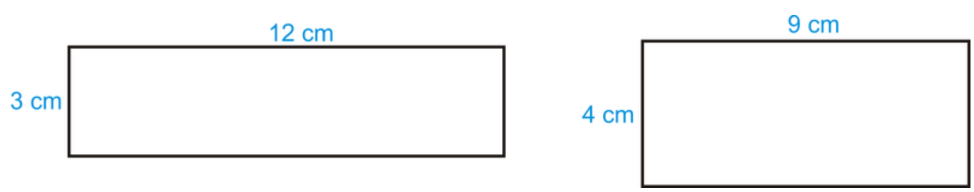
Congruent Areas Postulate: If two figures are congruent, they have the same area.

This postulate needs no proof because congruent figures have the same amount of space inside them. However, two figures with the same area are not necessarily congruent.

Example 5: Draw two different rectangles with an area of 36 cm^2.

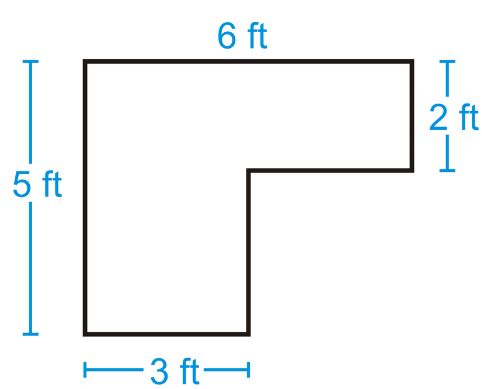
Solution: Think of all the different factors of 36. These can all be dimensions of the different rectangles.

Other possibilities could be $6 \times 6, 2 \times 18$, and 1×36.

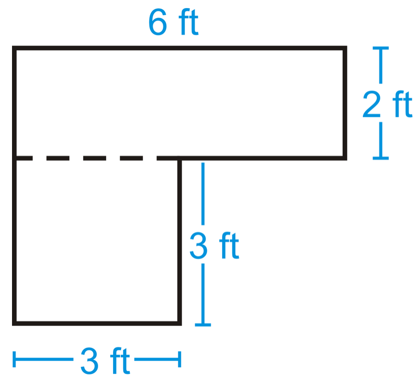


Area Addition Postulate: If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Example 6: Find the area of the figure below. You may assume all sides are perpendicular.



Solution: Split the shape into two rectangles and find the area of each.

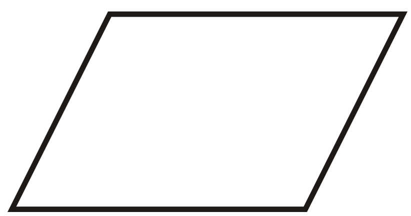


$$A_{top\ rectangle} = 6 \cdot 2 = 12\ ft^2$$
$$A_{bottom\ square} = 3 \cdot 3 = 9\ ft^2$$

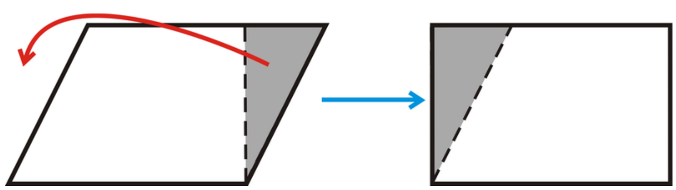
The total area is $12 + 9 = 21\ ft^2$.

Area of a Parallelogram

Recall that a parallelogram is a quadrilateral whose opposite sides are parallel.



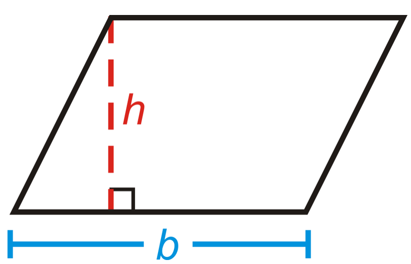
To find the area of a parallelogram, make it into a rectangle.



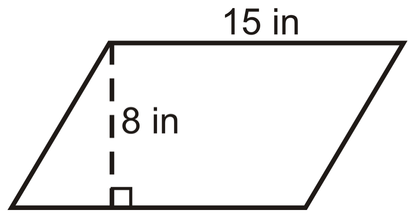
From this, we see that the area of a parallelogram is the same as the area of a rectangle.

Area of a Parallelogram: The area of a parallelogram is $A = bh$.

Be careful! The height of a parallelogram is always perpendicular to the base. This means that the sides are ***not*** the height.



Example 7: Find the area of the parallelogram.



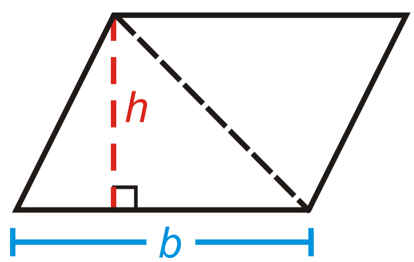
Solution: $A = 15 \cdot 8 = 120 \ in^2$

Example 8: If the area of a parallelogram is 56 $units^2$ and the base is 4 units, what is the height?

Solution: Plug in what we know to the area formula and solve for the height.

$$\begin{aligned} 56 &= 4h \\ 14 &= h \end{aligned}$$

Area of a Triangle

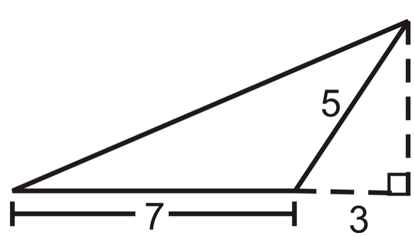


If we take parallelogram and cut it in half, along a diagonal, we would have two congruent triangles. Therefore, the formula for the area of a triangle is the same as the formula for area of a parallelogram, but cut in half.

Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$.

In the case that the triangle is a right triangle, then the height and base would be the legs of the right triangle. If the triangle is an obtuse triangle, the altitude, or height, could be outside of the triangle.

Example 9: Find the area and perimeter of the triangle.

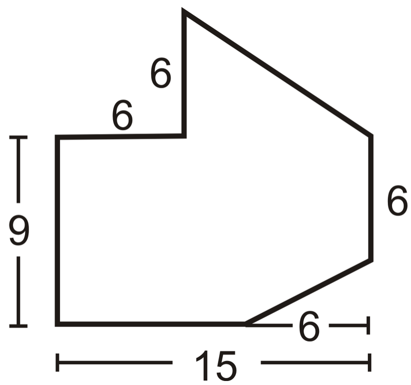


Solution: This is an obtuse triangle. First, to find the area, we need to find the height of the triangle. We are given the two sides of the small right triangle, where the hypotenuse is also the short side of the obtuse triangle. From these values, we see that the height is 4 because this is a 3-4-5 right triangle. The area is $A = \frac{1}{2}(4)(7) = 14 \ units^2$.

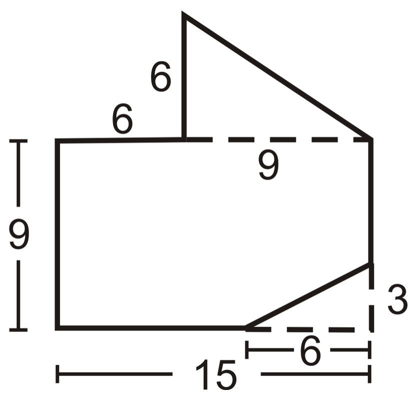
To find the perimeter, we would need to find the longest side of the obtuse triangle. If we used the dotted lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10. Use the Pythagorean Theorem.

$$\begin{aligned} 4^2+10^2 &= c^2 \\ 16+100 &= c^2 \\ c &= \sqrt{116} \approx 10.77 \qquad \text{The perimeter is } 7+5+10.77 = 22.77 \textit{ units} \end{aligned}$$

Example 10: Find the area of the figure below.



Solution: Divide the figure into a triangle and a rectangle with a small rectangle cut out of the lower right-hand corner.



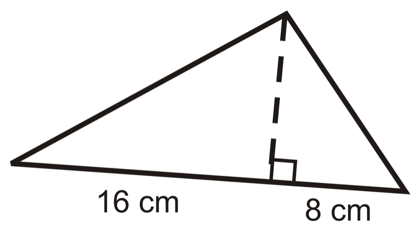
$$\begin{aligned} A &= A_{top\ triangle} + A_{rectangle} - A_{small\ triangle} \\ A &= \left(\frac{1}{2}\cdot 6\cdot 9\right) + (9\cdot 15) + \left(\frac{1}{2}\cdot 3\cdot 6\right) \\ A &= 27+135+9 \\ A &= 171 \textit{ units}^2 \end{aligned}$$

Know What? Revisited The area of an Eastern King is 6080 in^2 and the California King is 6048 in^2. The perimeter of both beds is 312 in. Because Ed is 6'4", he should probably get the California King because it is 4 inches longer.

Review Questions

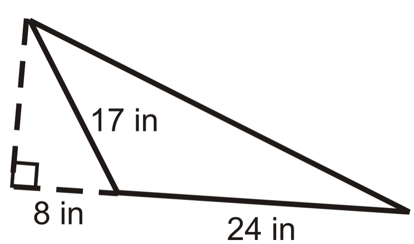
1. Find the area and perimeter of a square with sides of length 12 in.
2. Find the area and perimeter of a rectangle with height of 9 cm and base of 16 cm.
3. Find the area of a parallelogram with height of 20 m and base of 18 m.
4. Find the area and perimeter of a rectangle if the height is 8 and the base is 14.
5. Find the area and perimeter of a square if the sides are 18 ft.
6. If the area of a square is 81 ft^2, find the perimeter.
7. If the perimeter of a square is 24 in, find the area.
8. Find the area of a triangle with base of length 28 cm and height of 15 cm.
9. What is the height of a triangle with area 144 m^2 and a base of 24 m?
10. The perimeter of a rectangle is 32. Find two different dimensions that the rectangle could be.
11. Draw two different rectangles that haven an area of 90 mm^2.
12. Write the converse of the Congruent Areas Postulate. Determine if it is a true statement. If not, write a counterexample. If it is true, explain why.

Use the triangle to answer the following questions.



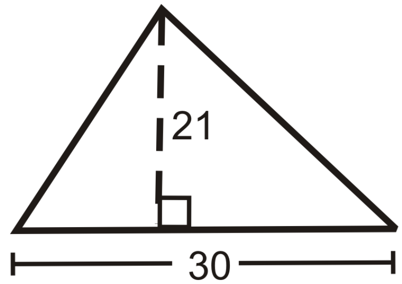
13. Find the height of the triangle by using the geometric mean.
14. Find the perimeter.
15. Find the area.

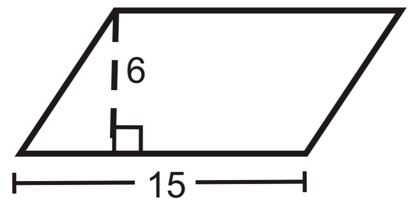
Use the triangle to answer the following questions.

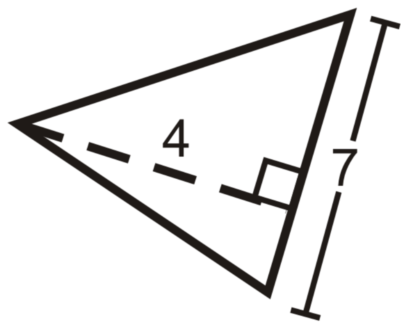


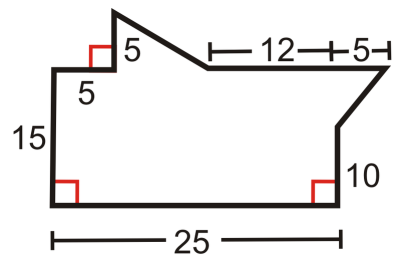
16. Find the height of the triangle.
17. Find the perimeter.
18. Find the area.

Find the area of the following shapes.

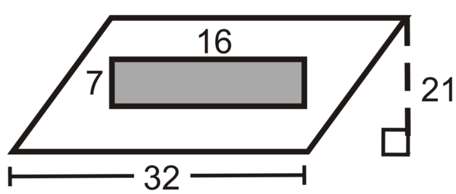
19. 

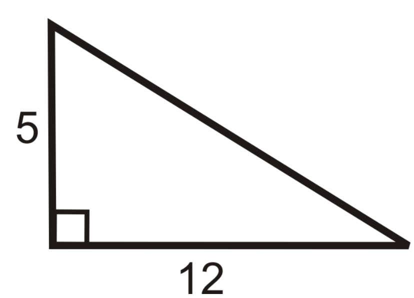
20. 

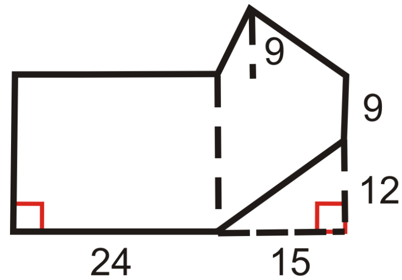
21. 

22. 

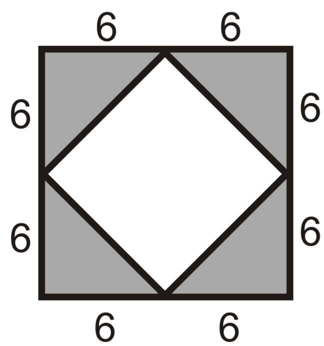
23. Find the area of the unshaded region.



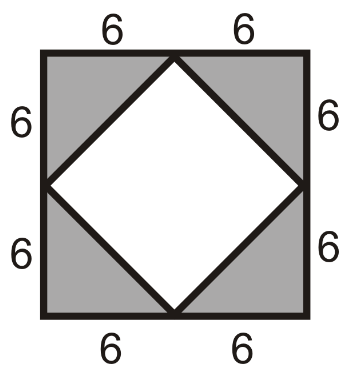
24. 

25. 

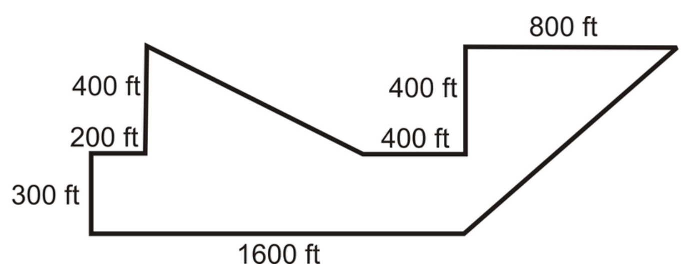
26. Find the area of the shaded region.



27. Find the area of the unshaded region.



28. Lin bought a tract of land for a new apartment complex. The drawing below shows the measurements of the sides of the tract. Approximately how many acres of land did Lin buy? You may assume any angles that look like right angles are 90°. (1 acre ≈ 40,000 square feet)



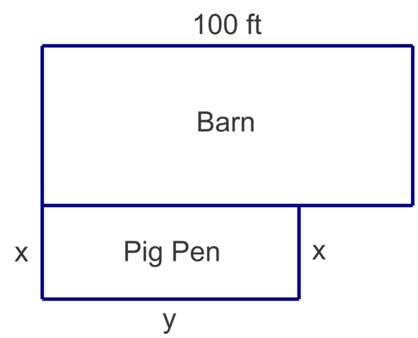
Challenge Problems

For problems 29 and 30 find the dimensions of the rectangles with the given information.

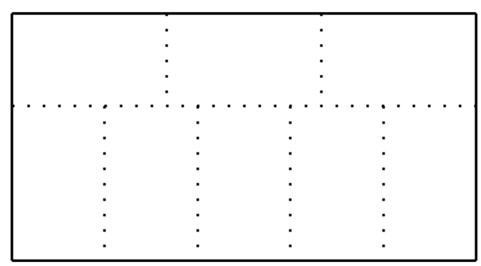
29. A rectangle with a perimeter of 20 units and an area of 24 $units^2$.
30. A rectangle with a perimeter of 72 units and an area of 288 $units^2$.

For problems 31 and 32 find the height and area of the equilateral triangle with the given perimeter.

31. Perimeter 18 units.
32. Perimeter 30 units.
33. Generalize your results from problems 31 and 32 into a formula to find the height and area of an equilateral triangle with side length x.
34. Linus has 100 ft of fencing to use in order to enclose a 1200 square foot rectangular pig pen. The pig pen is adjacent to the barn so he only needs to form three sides of the rectangular area as shown below. What dimensions should the pen be?



35. A rectangle with perimeter 138 units is divided into 8 congruent rectangles as shown in the diagram below. Find the perimeter and area of one of the 8 congruent rectangles.



Review Queue Answers

1. *Possible* Answers

Perimeter: The distance around a shape.

Area: The space inside a shape.

2. (a) $x = \pm 11$

(b) $x = \pm 2\sqrt{5}$

(c) $x = 4, 2$

3. $4+4+7+7 = 22$

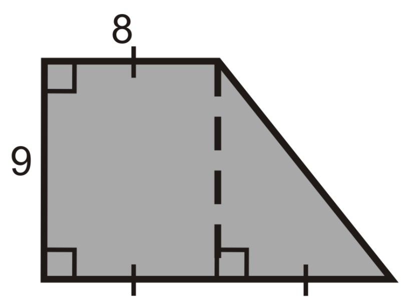
10.2 Trapezoids, Rhombi, and Kites

Learning Objectives

- Derive and use the area formulas for trapezoids, rhombi, and kites.

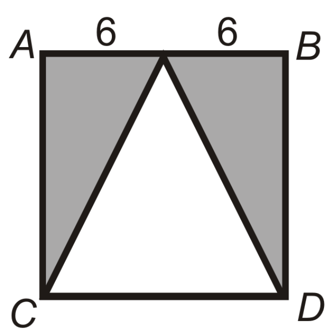
Review Queue

Find the area the *shaded* regions in the figures below.

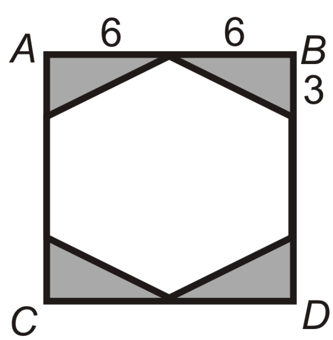


a.

b. *ABCD* is a square.



c. *ABCD* is a square.



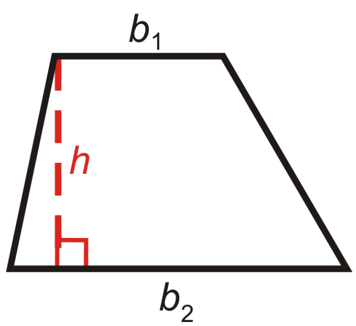
d. Find the area of #1 using a different method.

Know What? The Brazilian flag is to the right. The flag has dimensions of 20×14 (units vary depending on the size, so we will not use any here). The vertices of the yellow rhombus in the middle are 1.7 units from the midpoint of each side.

Find the total area of the flag and the area of the rhombus (including the circle). *Do not round your answers.*

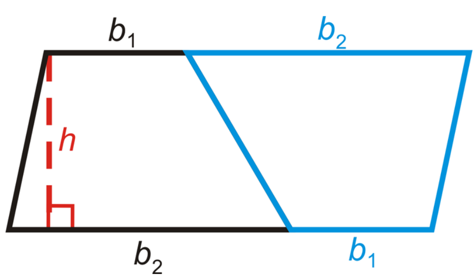
Area of a Trapezoid

Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases. The perpendicular distance between the parallel sides is the height, or altitude, of the trapezoid.



To find the area of the trapezoid, let's turn it into a parallelogram. To do this, make a copy of the trapezoid and then rotate the copy 180°.

Now, this is a parallelogram with height h and base $b_1 + b_2$. Let's find the area of this shape.



$$A = h(b_1 + b_2)$$

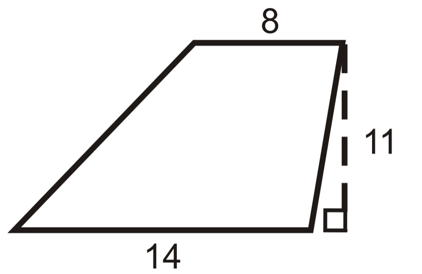
Because the area of this parallelogram is made up of two congruent trapezoids, the area of one trapezoid would be $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Trapezoid: The area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$.

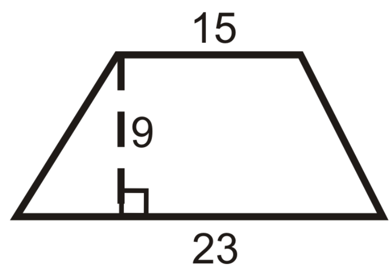
The formula for the area of a trapezoid could also be written as *the average of the bases time the height.*

Example 1: Find the area of the trapezoids below.

a)



b)

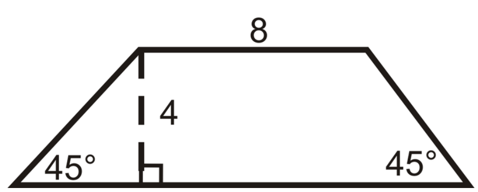


Solution:

a) $A = \frac{1}{2}(11)(14+8)$
$A = \frac{1}{2}(11)(22)$
$A = 121\ units^2$

b) $A = \frac{1}{2}(9)(15+23)$
$A = \frac{1}{2}(9)(38)$
$A = 171\ units^2$

Example 2: Find the perimeter and area of the trapezoid.



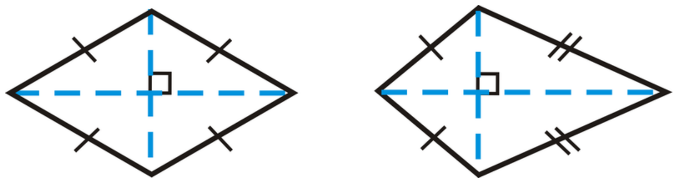
Solution: Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

$$P = 8 + 4\sqrt{2} + 16 + 4\sqrt{2} \qquad A = \frac{1}{2}(4)(8+16)$$
$$P = 24 + 8\sqrt{2} \approx 35.3\ units \qquad A = 48\ units^2$$

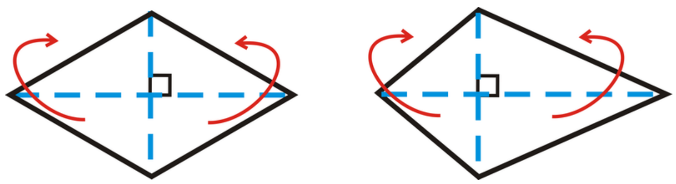
Area of a Rhombus and Kite

Recall that a rhombus is an equilateral quadrilateral and a kite has adjacent congruent sides.

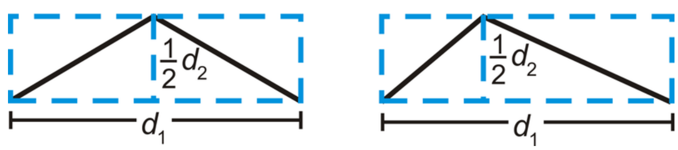
Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.



Notice that the diagonals divide each quadrilateral into 4 triangles. In the rhombus, all 4 triangles are congruent and in the kite there are two sets of congruent triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.



So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.



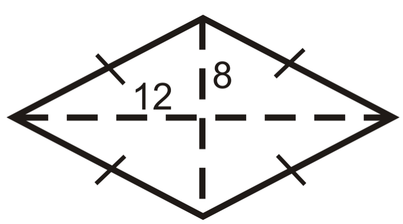
Area of a Rhombus: If the diagonals of a rhombus are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

Area of a Kite: If the diagonals of a kite are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

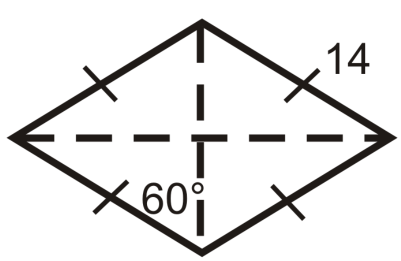
You could also say that the area of a kite and rhombus are *half the product of the diagonals.*

Example 3: Find the perimeter and area of the rhombi below.

a)



b)



Solution: In a rhombus, all four triangles created by the diagonals are congruent.

a) To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

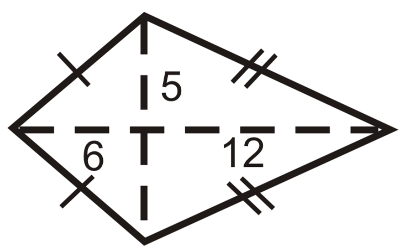
$$\begin{aligned} 12^2+8^2 &= side^2 \\ 144+64 &= side^2 \\ side &= \sqrt{208}=4\sqrt{13} \\ P &= 4\left(4\sqrt{13}\right)=16\sqrt{13} \end{aligned} \qquad \begin{aligned} A &= \frac{1}{2}\cdot 16\cdot 24 \\ A &= 192 \end{aligned}$$

b) Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is $7\sqrt{3}$.

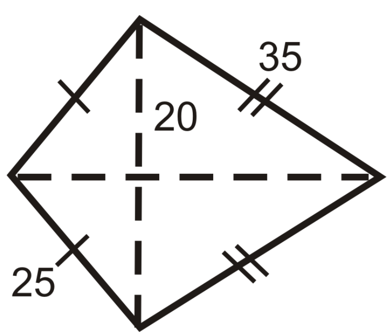
$$P=4\cdot 14=56 \qquad A=\frac{1}{2}\cdot 7\cdot 7\sqrt{3}=\frac{49\sqrt{3}}{2}\approx 42.44$$

Example 4: Find the perimeter and area of the kites below.

a)



b)



Solution: In a kite, there are two pairs of congruent triangles. You will need to use the Pythagorean Theorem in both problems to find the length of sides or diagonals.

a)

Shorter sides of kite

$$6^2+5^2=s_1^2$$
$$36+25=s_1^2$$
$$s_1=\sqrt{61}$$

Longer sides of kite

$$12^2+5^2=s_2^2$$
$$144+25=s_2^2$$
$$s_2=\sqrt{169}=13$$

$$P=2\left(\sqrt{61}\right)+2(13)=2\sqrt{61}+26\approx 41.6$$
$$A=\frac{1}{2}(10)(18)=90$$

b)

Smaller diagonal portion

$$20^2+d_s^2=25^2$$
$$d_s^2=225$$
$$d_s=15$$

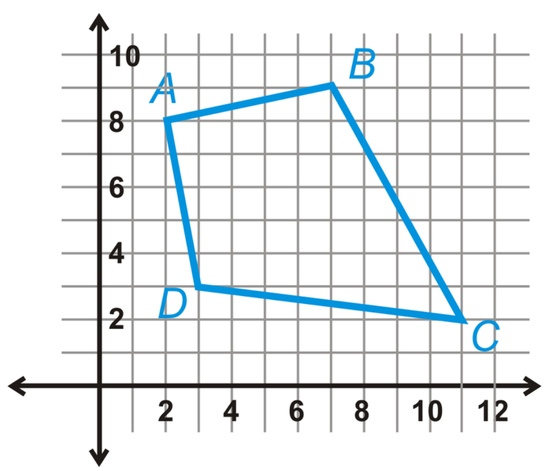
Larger diagonal portion

$$20^2+d_l^2=35^2$$
$$d_l^2=825$$
$$d_l=5\sqrt{33}$$

$$P=2(25)+2(35)=120$$
$$A=\frac{1}{2}\left(15+5\sqrt{33}\right)(40)\approx 874.5$$

Example 5: The vertices of a quadrilateral are $A(2,8), B(7,9), C(11,2)$, and $D(3,3)$. Determine the type of quadrilateral and find its area.

Solution: For this problem, it might be helpful to plot the points. From the graph we can see this is probably a kite. Upon further review of the sides, $AB = AD$ and $BC = DC$ (you can do the distance formula to verify). Let's see if the diagonals are perpendicular by calculating their slopes.



$$m_{AC}=\frac{2-8}{11-2}=-\frac{6}{9}=-\frac{2}{3}$$
$$m_{BD}=\frac{9-3}{7-3}=\frac{6}{4}=\frac{3}{2}$$

Yes, the diagonals are perpendicular because the slopes are opposite signs and reciprocals. *ABCD* is a kite. To find the area, we need to find the length of the diagonals. Use the distance formula.

$$
\begin{aligned}
d_1 &= \sqrt{(2-11)^2+(8-2)^2} \\
&= \sqrt{(-9)^2+6^2} \\
&= \sqrt{81+36} = \sqrt{117} = 3\sqrt{13}
\end{aligned}
\qquad
\begin{aligned}
d_2 &= \sqrt{(7-3)^2+(9-3)^2} \\
&= \sqrt{4^2+6^2} \\
&= \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}
\end{aligned}
$$

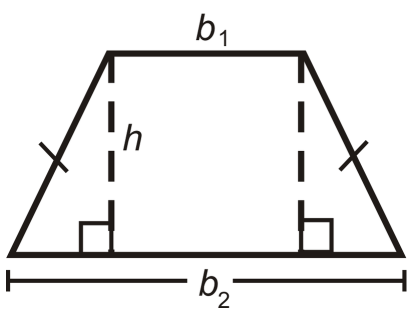
Now, plug these lengths into the area formula for a kite.

$$A = \frac{1}{2}\left(3\sqrt{13}\right)\left(2\sqrt{13}\right) = 39\ units^2$$

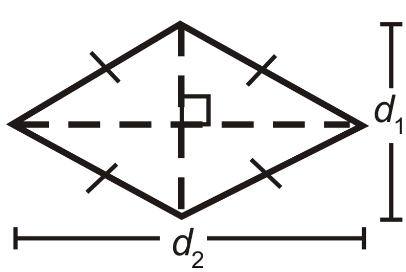
Know What? Revisited The total area of the Brazilian flag is $A = 14 \cdot 20 = 280\ units^2$. To find the area of the rhombus, we need to find the length of the diagonals. One diagonal is $20 - 1.7 - 1.7 = 16.6\ units$ and the other is $14 - 1.7 - 1.7 = 10.6\ units$. The area is $A = \frac{1}{2}(16.6)(10.6) = 87.98\ units^2$.

Review Questions

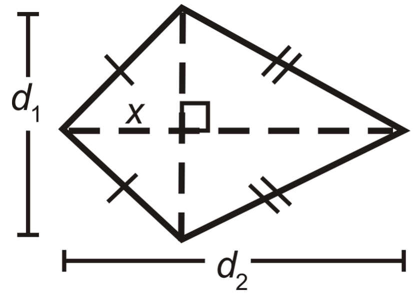
1. Do you think all rhombi and kites with the same diagonal lengths have the same area? *Explain* your answer.
2. Use the isosceles trapezoid to show that the area of this trapezoid can also be written as the sum of the area of the two triangles and the rectangle in the middle. Write the formula and then reduce it to equal $\frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$.



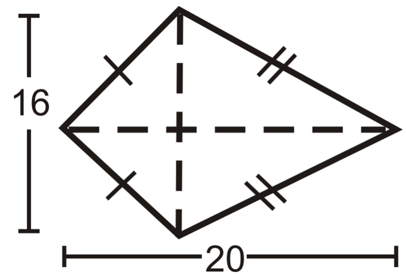
3. Use this picture of a rhombus to show that the area of a rhombus is equal to the sum of the areas of the four congruent triangles. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.

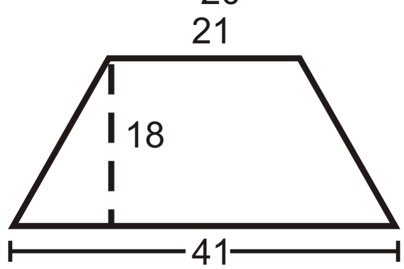


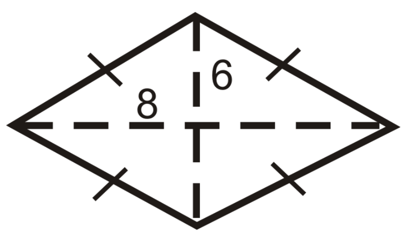
4. Use this picture of a kite to show that the area of a kite is equal to the sum of the areas of the two pairs of congruent triangles. Recall that d_1 is bisected by d_2. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.

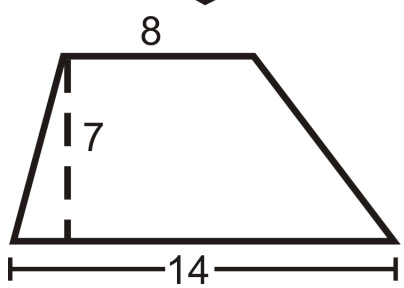


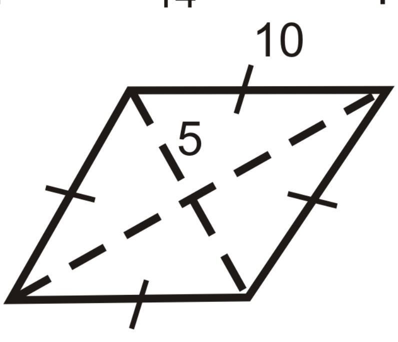
Find the area of the following shapes. *Leave answers in simplest radical form.*

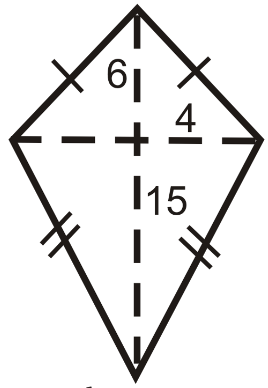
5. 

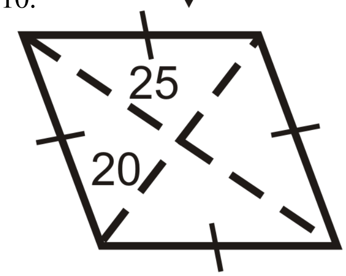
6. 

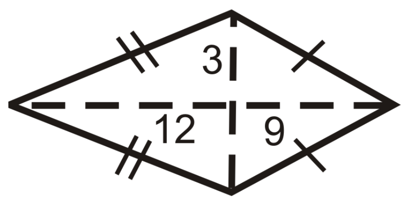
7. 

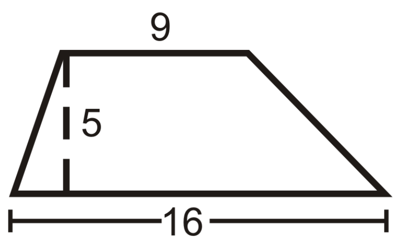
8. 

9. 

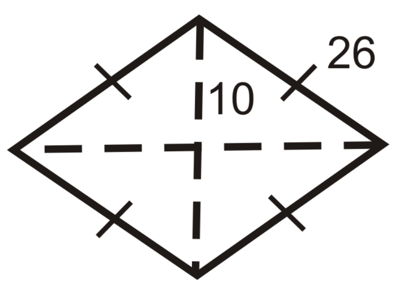
10. 

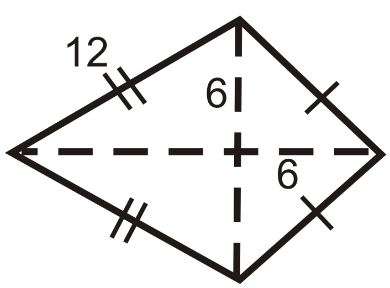
11. 

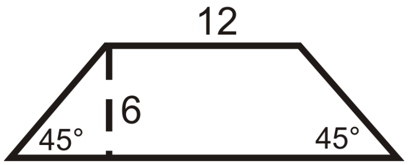
12. 

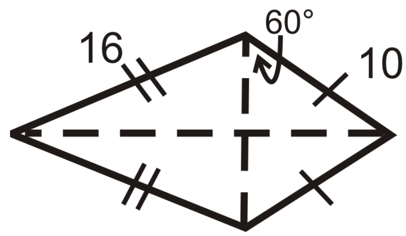
13. 

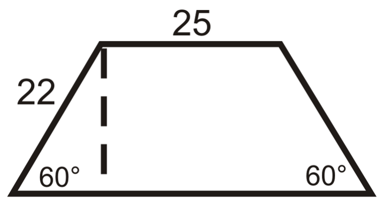
Find the area and perimeter of the following shapes. *Leave answers in simplest radical form.*

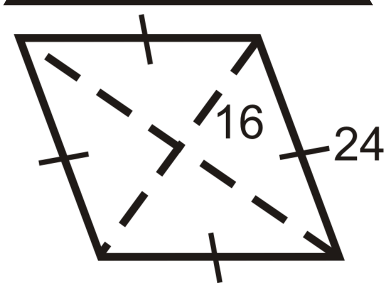
14. 

15. 

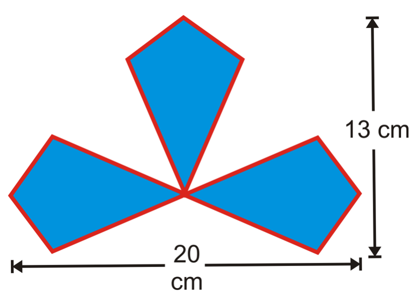
16.


17.


18.


19.


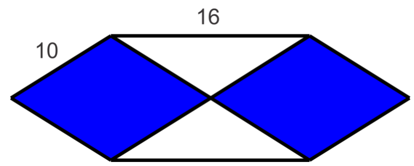
20. Quadrilateral $ABCD$ has vertices $A(-2,0), B(0,2), C(4,2)$, and $D(0,-2)$. Show that $ABCD$ is a trapezoid and find its area. *Leave your answer in simplest radical form.*
21. Quadrilateral $EFGH$ has vertices $E(2,-1), F(6,-4), G(2,-7)$, and $H(-2,-4)$. Show that $EFGH$ is a rhombus and find its area.
22. The area of a rhombus is 32 $units^2$. What are two possibilities for the lengths of the diagonals?
23. The area of a kite is 54 $units^2$. What are two possibilities for the lengths of the diagonals?
24. Sherry designed the logo for a new company. She used three congruent kites. What is the area of the entire logo?



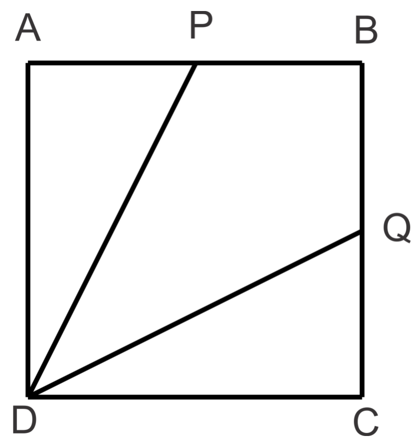
For problems 25-27, determine what kind of quadrilateral $ABCD$ is and find its area.

25. $A(-2,3), B(2,3), C(4,-3), D(-2,-1)$
26. $A(0,1), B(2,6), C(8,6), D(13,1)$
27. $A(-2,2), B(5,6), C(6,-2), D(-1,-6)$
28. Given that the lengths of the diagonals of a kite are in the ratio 4:7 and the area of the kite is 56 square units, find the lengths of the diagonals.
29. Given that the lengths of the diagonals of a rhombus are in the ratio 3:4 and the area of the rhombus is 54 square units, find the lengths of the diagonals.

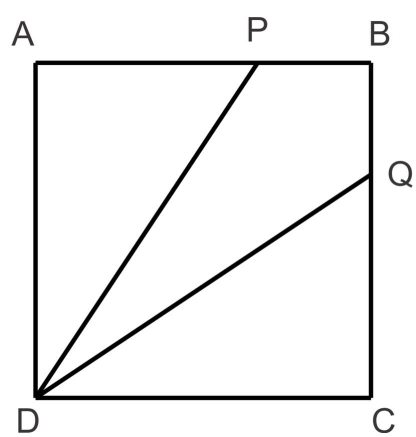
30. Sasha drew this plan for a wood inlay he is making. 10 is the length of the slanted side and 16 is the length of the horizontal line segment as shown in the diagram. Each shaded section is a rhombus. What is the total area of the shaded sections?



31. In the figure to the right, $ABCD$ is a square. $AP = PB = BQ$ and $DC = 20\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



32. In the figure to the right, $ABCD$ is a square. $AP = 20\ ft$ and $PB = BQ = 10\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



Review Queue Answers

a. $A = 9(8) + \left[\frac{1}{2}(9)(8)\right] = 72 + 36 = 108\ units^2$
b. $A = \frac{1}{2}(6)(12)2 = 72\ units^2$
c. $A = 4\left[\frac{1}{2}(6)(3)\right] = 36\ units^2$
d. $A = 9(16) - \left[\frac{1}{2}(9)(8)\right] = 144 - 36 = 108\ units^2$

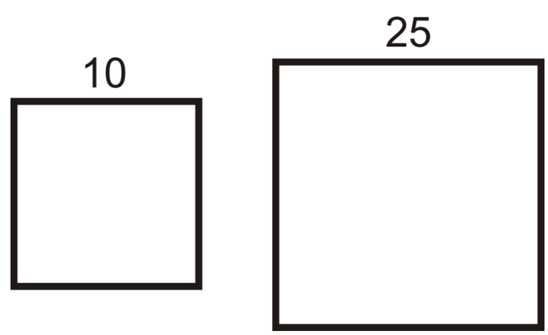
10.3 Areas of Similar Polygons

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas.
- Apply scale factors to solve problems about areas of similar polygons.

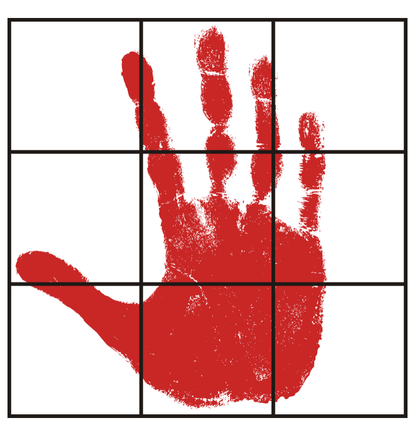
Review Queue

a. Are two squares similar? Are two rectangles?



b. Find the scale factor of the sides of the similar shapes. Both figures are squares.
c. Find the area of each square.
d. Find the ratio of the smaller square's area to the larger square's area. Reduce it. How does it relate to the scale factor?

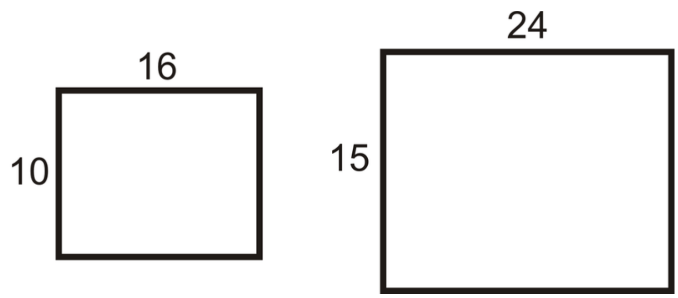
Know What? One use of scale factors and areas is scale drawings. This technique takes a small object, like the handprint to the right, divides it up into smaller squares and then blows up the individual squares. In this Know What? you are going to make a scale drawing of your own hand. Either trace your hand or stamp it on a piece of paper. Then, divide your hand into 9 squares, like the one to the right, probably 2 $in \times 2$ in. Take a larger piece of paper and blow up each square to be 6 $in \times 6$ in (meaning you need at least an 18 in square piece of paper). Once you have your 6 $in \times 6$ in squares drawn, use the proportions and area to draw in your enlarged handprint.



Areas of Similar Polygons

In Chapter 7, we learned about similar polygons. Polygons are similar when the corresponding angles are equal and the corresponding sides are in the same proportion. In that chapter we also discussed the relationship of the perimeters of similar polygons. Namely, the scale factor for the sides of two similar polygons is the same as the ratio of the perimeters.

Example 1: The two rectangles below are similar. Find the scale factor and the ratio of the perimeters.



Solution: The scale factor is $\frac{16}{24}$, which reduces to $\frac{2}{3}$. The perimeter of the smaller rectangle is 52 units. The perimeter of the larger rectangle is 78 units. The ratio of the perimeters is $\frac{52}{78} = \frac{2}{3}$.

The ratio of the perimeters is the same as the scale factor. In fact, the ratio of any part of two similar shapes (diagonals, medians, midsegments, altitudes, etc.) is the same as the scale factor.

Example 2: Find the area of each rectangle from Example 1. Then, find the ratio of the areas.

Solution:

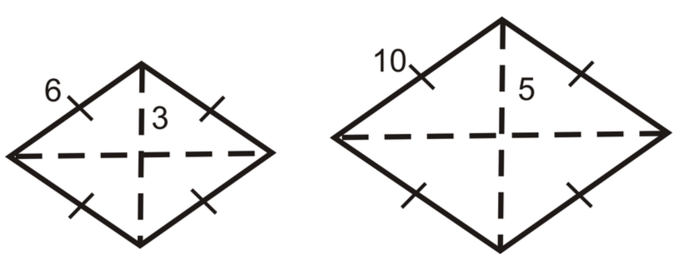
$$A_{small} = 10 \cdot 16 = 160 \; units^2$$
$$A_{large} = 15 \cdot 24 = 360 \; units^2$$

The ratio of the areas would be $\frac{160}{360} = \frac{4}{9}$.

The ratio of the sides, or scale factor was $\frac{2}{3}$ and the ratio of the areas is $\frac{4}{9}$. Notice that the ratio of the areas is the ***square*** of the scale factor. An easy way to remember this is to think about the units of area, which are always ***squared.*** Therefore, you would always ***square*** the scale factor to get the ratio of the areas.

Area of Similar Polygons Theorem: If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

Example 2: Find the ratio of the areas of the rhombi below. The rhombi are similar.



Solution: There are two ways to approach this problem. One way would be to use the Pythagorean Theorem to find the length of the 3^{rd} side in the triangle and then apply the area formulas and make a ratio. The second, and easier way, would be to find the ratio of the sides and then square that. $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$

Example 3: Two trapezoids are similar. If the scale factor is $\frac{3}{4}$ and the area of the smaller trapezoid is 81 cm^2, what is the area of the larger trapezoid?

Solution: First, the ratio of the areas would be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Now, we need the area of the larger trapezoid. To find this, we would multiply the area of the smaller trapezoid by the scale factor. However, we would need to flip the scale factor over to be $\frac{16}{9}$ because we want the larger area. This means we need to multiply by a scale factor that is larger than one. $A = \frac{16}{9} \cdot 81 = 144\ cm^2$.

Example 4: Two triangles are similar. The ratio of the areas is $\frac{25}{64}$. What is the scale factor?

Solution: The scale factor is $\sqrt{\frac{25}{64}} = \frac{5}{8}$.

Example 5: Using the ratios from Example 3, find the length of the base of the smaller triangle if the length of the base of the larger triangle is 24 units.

Solution: All you would need to do is multiply the scale factor we found in Example 3 by 24.

$$b = \frac{5}{8} \cdot 24 = 15\ units$$

Know What? Revisited You should end up with an 18 *in* × 18 *in* drawing of your handprint.

Review Questions

Determine the ratio of the areas, given the ratio of the sides of a polygon.

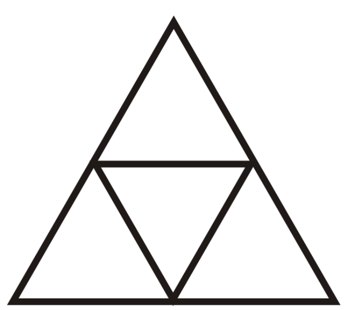
1. $\frac{3}{5}$
2. $\frac{1}{4}$
3. $\frac{7}{2}$
4. $\frac{6}{11}$

Determine the ratio of the sides of a polygon, given the ratio of the areas.

5. $\frac{1}{36}$
6. $\frac{4}{81}$
7. $\frac{49}{9}$
8. $\frac{25}{144}$

This is an equilateral triangle made up of 4 congruent equilateral triangles.

9. What is the ratio of the areas of the large triangle to one of the small triangles?



10. What is the scale factor of large to small triangle?
11. If the area of the large triangle is 20 $units^2$, what is the area of a small triangle?
12. If the length of the altitude of a small triangle is $2\sqrt{3}$, find the perimeter of the large triangle.
13. Carol drew two equilateral triangles. Each side of one triangle is 2.5 times as long as a side of the other triangle. The perimeter of the smaller triangle is 40 cm. What is the perimeter of the larger triangle?
14. If the area of the smaller triangle is 75 cm^2, what is the area of the larger triangle from #13?
15. Two rectangles are similar with a scale factor of $\frac{4}{7}$. If the area of the larger rectangle is 294 in^2, find the area of the smaller rectangle.
16. Two triangles are similar with a scale factor of $\frac{1}{3}$. If the area of the smaller triangle is 22 ft^2, find the area of the larger triangle.
17. The ratio of the areas of two similar squares is $\frac{16}{81}$. If the length of a side of the smaller square is 24 units, find the length of a side in the larger square.
18. The ratio of the areas of two right triangles is $\frac{2}{3}$. If the length of the hypotenuse of the larger triangle is 48 units, find the length of the smaller triangle's hypotenuse.

Questions 19-22 build off of each other. You may assume the problems are connected.

19. Two similar rhombi have areas of 72 $units^2$ and 162 $units^2$. Find the ratio of the areas.
20. Find the scale factor.
21. The diagonals in these rhombi are congruent. Find the length of the diagonals and the sides.
22. What type of rhombi are these quadrilaterals?
23. The area of one square on a game board is exactly twice the area of another square. Each side of the larger square is 50 mm long. How long is each side of the smaller square?
24. The distance from Charleston to Morgantown is 160 miles. The distance from Fairmont to Elkins is 75 miles. Charleston and Morgantown are 5 inches apart on a map. How far apart are Fairmont and Elkins on the same map.
25. Marlee is making models of historic locomotives (train engines). She uses the same scale for all of her models. The S1 locomotive was 140 ft long. The model is 8.75 inches long. The 520 Class locomotive was 87 feet long. What is the scale of Marlee's models? How long is the model of the 520 Class locomotive?
26. Tommy is drawing a floor plan for his dream home. On his drawing, 1cm represents 2 ft of the actual home. The actual dimensions of the dream home are 55 ft by 40 ft. What will the dimensions of his floor plan be? Will his scale drawing fit on a standard 8.5 in by 11 in piece of paper? Justify your answer.
27. Anne wants to purchase advertisement space in the school newspaper. Each square inch of advertisement space sells for $3.00. She wants to purchase a rectangular space with length and width in the ratio 3:2 and she has up to $50.00 to spend. What are the dimensions of the largest advertisement she can afford to purchase?
28. Aaron wants to enlarge a family photo from a 5 by 7 print to a print with an area of 140 inches. What are the dimensions of this new photo?
29. A popular pizza joint offers square pizzas: Baby Bella pizza with 10 inch sides, the Mama Mia pizza with 14 inch sides and the Big Daddy pizza with 18 inch sides. If the prices for these pizzas are $5.00, $9.00 and $15.00 respectively, find the price per square inch of each pizza. Which is the best deal?
30. Krista has a rectangular garden with dimensions 2 ft by 3 ft. She uses $\frac{2}{3}$ of a bottle of fertilizer to cover this area. Her friend, Hadleigh, has a garden with dimensions that are 1.5 times as long. How many bottles of fertilizer will she need?

Review Queue Answers

a. Two squares are always similar. Two rectangles can be similar as long as the sides are in the same proportion.
b. $\frac{10}{25} = \frac{2}{5}$
c. $A_{small} = 100, A_{large} = 625$

d. $\frac{100}{625} = \frac{4}{25}$, this is the square of the scale factor.

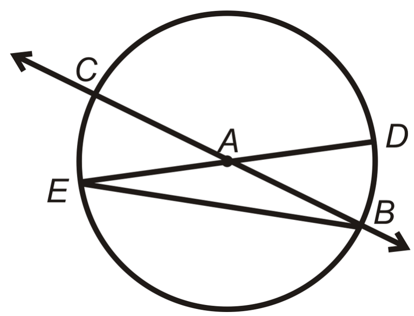
10.4 Circumference and Arc Length

Learning Objectives

- Find the circumference of a circle.
- Define the length of an arc and find arc length.

Review Queue

a. Find a central angle in that intercepts $\widehat{CE}$



b. Find an inscribed angle that intercepts $\widehat{CE}$.
c. How many degrees are in a circle? Find $m\widehat{ECD}$.
d. If $m\widehat{CE} = 26^\circ$, find $m\widehat{CD}$ and $m\angle CBE$.

Know What? A typical large pizza has a diameter of 14 inches and is cut into 8 or 10 pieces. Think of the crust as the circumference of the pizza. Find the *length* of the crust for the entire pizza. Then, find the length of the crust for one piece of pizza if the entire pizza is cut into a) 8 pieces or b) 10 pieces.

Circumference of a Circle

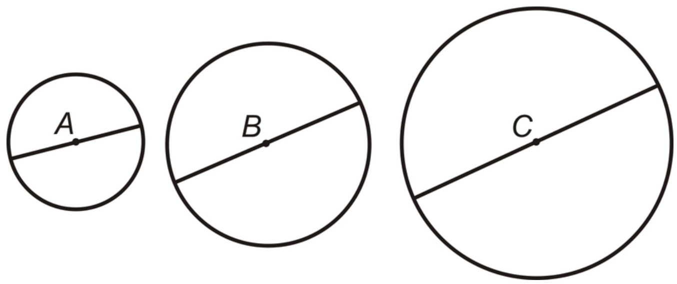
Circumference: The distance around a circle.

The circumference can also be called the perimeter of a circle. However, we use the term circumference for circles because they are round. The term perimeter is reserved for figures with straight sides. In order to find the formula for the circumference of a circle, we first need to determine the ratio between the circumference and diameter of a circle.

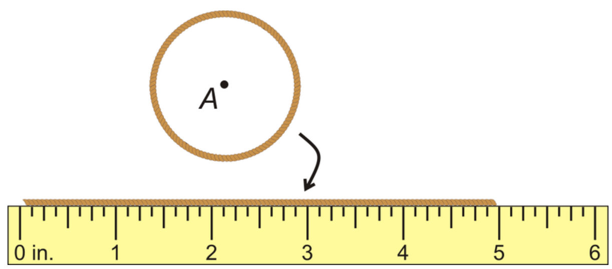
Investigation 10-1: Finding π (pi)

Tools Needed: paper, pencil, compass, ruler, string, and scissors

a. Draw three circles with radii of 2 in, 3 in, and 4 in. Label the centers of each A, B, and C.
b. Draw in the diameters and determine their lengths. Are all the diameter lengths the same in $\odot A$? $\odot B$? $\odot C$?



c. Take the string and outline each circle with it. The string represents the circumference of the circle. Cut the string so that it perfectly outlines the circle. Then, lay it out straight and measure, in inches. Round your answer to the nearest $\frac{1}{8}$-inch. Repeat this for the other two circles.



d. Find $\frac{circumference}{diameter}$ for each circle. Record your answers to the nearest thousandth. What do you notice?

From this investigation, you should see that $\frac{circumference}{diameter}$ approaches 3.14159... The bigger the diameter, the closer the ratio was to this number. We call this number π, the Greek letter "pi." It is an irrational number because the decimal never repeats itself. Pi has been calculated out to the millionth place and there is still no pattern in the sequence of numbers. When finding the circumference and area of circles, we must use π.

π, or **"pi":** The ratio of the circumference of a circle to its diameter. It is approximately equal to 3.14159265358979323846...

To see more digits of π, go to http://www.eveandersson.com/pi/digits/ .

You are probably familiar with the formula for circumference. From Investigation 10-1, we found that $\frac{circumference}{diameter} = \pi$. Let's shorten this up and solve for the circumference.

$\frac{C}{d} = \pi$, multiplying both sides by d, we have $C = \pi d$. We can also say $C = 2\pi r$ because $d = 2r$.

Circumference Formula: If d is the diameter or r is the radius of a circle, then $C = \pi d$ or $C = 2\pi r$.

Example 1: Find the circumference of a circle with a radius of 7 cm.

Solution: Plug the radius into the formula.

$$C = 2\pi(7) = 14\pi \approx 44\ cm$$

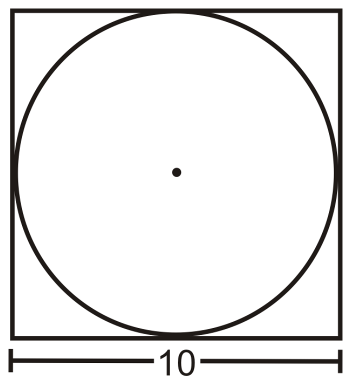
Depending on the directions in a given problem, you can either leave the answer in terms of π or multiply it out and get an approximation. Make sure you read the directions.

Example 2: The circumference of a circle is 64π. Find the diameter.

Solution: Again, you can plug in what you know into the circumference formula and solve for d.

$$64\pi = \pi d = 14\pi$$

Example 3: A circle is inscribed in a square with 10 in. sides. What is the circumference of the circle? Leave your answer in terms of π.



Solution: From the picture, we can see that the diameter of the circle is equal to the length of a side. Use the circumference formula.

$$C = 10\pi \; in.$$

Example 4: Find the perimeter of the square. Is it more or less than the circumference of the circle? Why?

Solution: The perimeter is $P = 4(10) = 40 \; in$. In order to compare the perimeter with the circumference we should change the circumference into a decimal.

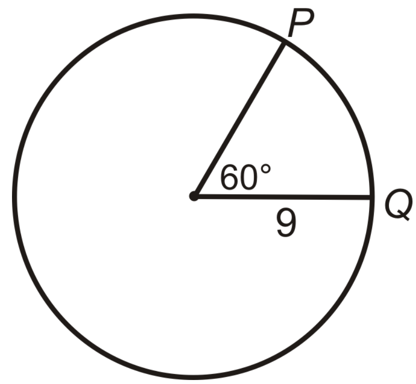
$C = 10\pi \approx 31.42 \; in$. This is less than the perimeter of the square, which makes sense because the circle is smaller than the square.

Arc Length

In Chapter 9, we measured arcs in degrees. This was called the "arc measure" or "degree measure." Arcs can also be measured in length, as a portion of the circumference.

Arc Length: The length of an arc or a portion of a circle's circumference.

The arc length is directly related to the degree arc measure. Let's look at an example.



Example 5: Find the length of $\widehat{PQ}$. Leave your answer in terms of π.

Solution: In the picture, the central angle that corresponds with $\widehat{PQ}$ is 60°. This means that $m\widehat{PQ} = 60^\circ$ as well. So, think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that $\left(\frac{60^\circ}{360^\circ} = \frac{1}{6}\right)$. Therefore, the length of $\widehat{PQ}$ is $\frac{1}{6}$ of the circumference.

$$length\ of\ \widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$$

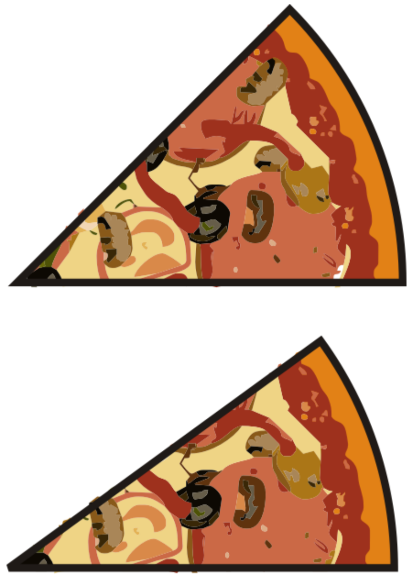
Arc Length Formula: If d is the diameter or r is the radius, the length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Example 6: The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

Solution: If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r.

$$\begin{aligned} 24\pi &= 2\pi r \\ 12 &= r \end{aligned}$$

Know What? Revisited The entire length of the crust, or the circumference of the pizza is $14\pi \approx 44$ *in*. In the picture to the right, the top piece of pizza is if it is cut into 8 pieces. Therefore, for $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ *inches* of crust. The bottom piece of pizza is if the pizza is cut into 10 pieces. For $\frac{1}{10}$ of the crust, one piece would have $\frac{44}{10} \approx 4.4$ *inches* of crust.



Review Questions

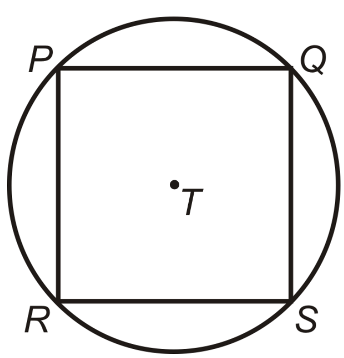
Fill in the following table. Leave all answers in terms of π.

TABLE 10.1:

	diameter	radius	circumference
1.	15		
2.		4	
3.	6		
4.			84π
5.		9	
6.			25π
7.			2π
8.	36		

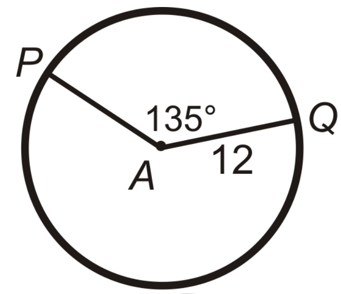
9. Find the radius of circle with circumference 88 in.
10. Find the circumference of a circle with $d = \frac{20}{\pi} cm$.

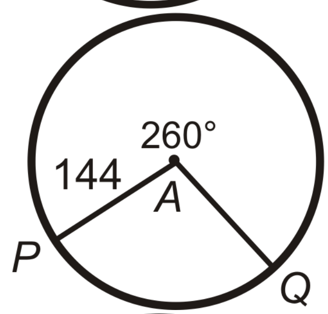
Square $PQSR$ is inscribed in $\odot T$. $RS = 8\sqrt{2}$.

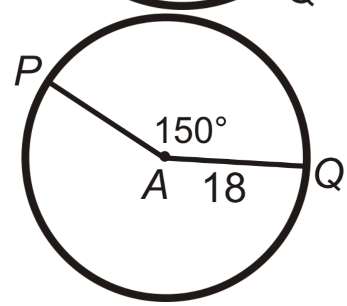


11. Find the length of the diameter of $\odot T$.
12. How does the diameter relate to $PQSR$?
13. Find the perimeter of $PQSR$.
14. Find the circumference of $\odot T$.

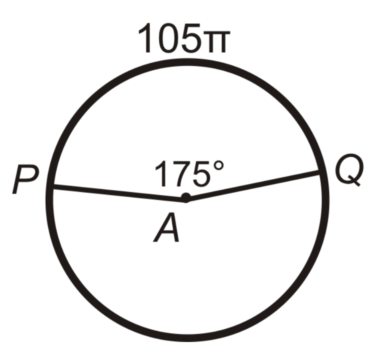
Find the arc length of $\widehat{PQ}$ in $\odot A$. Leave your answers in terms of π.

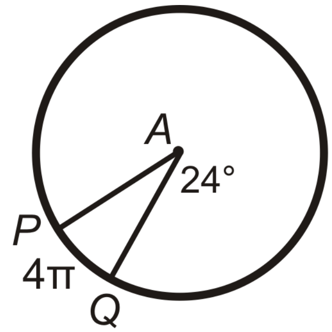
15. 

16. 

17. 

Find PA (the radius) in $\odot A$. Leave your answer in terms of π.

18. 



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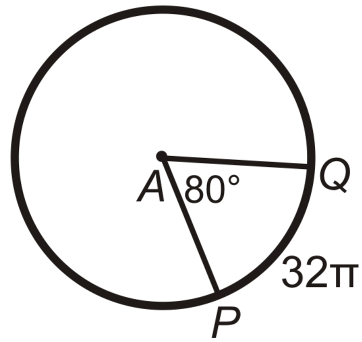
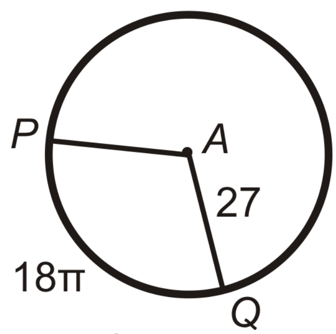


20.

Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.



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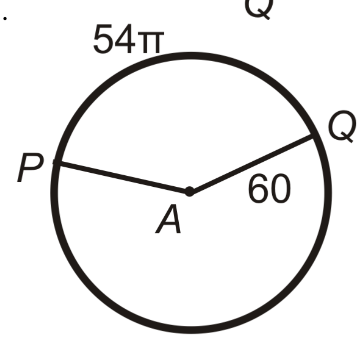

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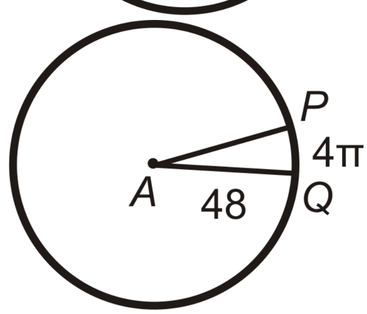
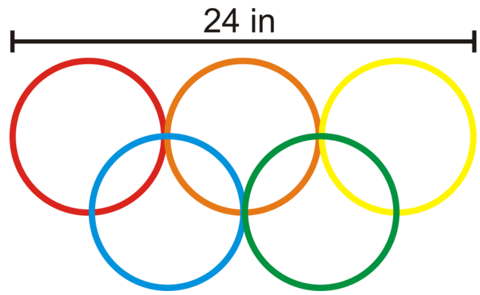


23.

24. The Olympics symbol is five congruent circles arranged as shown below. Assume the top three circles are tangent to each other. Brad is tracing the entire symbol for a poster. How far will his pen point travel?



25. A truck has tires with a 26 in diameter.
    a. How far does the truck travel every time a tire turns exactly once?
    b. How many times will the tire turn after the truck travels 1 mile? (1 mile = 5280 feet)
26. Mario's Pizza Palace offers a stuffed crust pizza in three sizes (diameter length) for the indicated prices: The Little Cheese, 8 in, $7.00 The Big Cheese, 10 in, $9.00 The Cheese Monster, 12 in, $12.00 What is the crust (in) to price ($) ratio for each of these pizzas? Michael thinks the cheesy crust is the best part of the pizza and wants to get the most crust for his money. Which pizza should he buy?
27. Jay is decorating a cake for a friend's birthday. They want to put gumdrops around the edge of the cake which has a 12 in diameter. Each gumdrop is has a diameter of 1.25 cm. To the nearest gumdrop, how many will they need?
28. A spedometer in a car measures the distance travelled by counting the rotations of the tires. The number of rotations required to travel one tenth of a mile in a particular vehicle is approximately 9.34. To the nearest inch, find the diameter of the wheel. (1 mile = 5280 feet)
29. Bob wants to put new weather stripping around a semicircular window above his door. The base of the window (diameter) is 36 inches. How much weather stripping does he need?
30. Each car on a Ferris wheel travels 942.5 ft during the 10 rotations of each ride. How high is each car at the highest point of each rotation?

Review Queue Answers

a. $\angle CAE$
b. $\angle CBE$
c. $360^\circ, 180^\circ$
d. $m\widehat{CD} = 180^\circ - 26^\circ = 154^\circ, m\angle CBE = 13^\circ$

10.5 Areas of Circles and Sectors

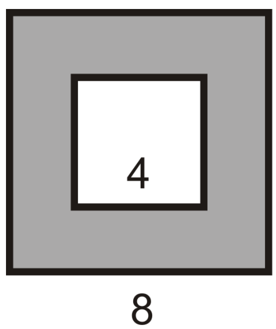
Learning Objectives

- Find the area of circles, sectors, and segments.

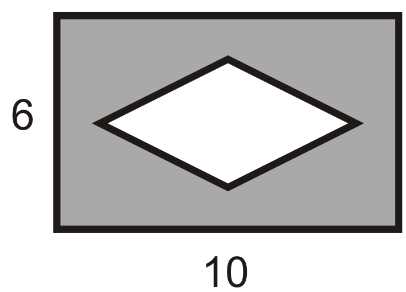
Review Queue

Find the area of the shaded region in the following figures.

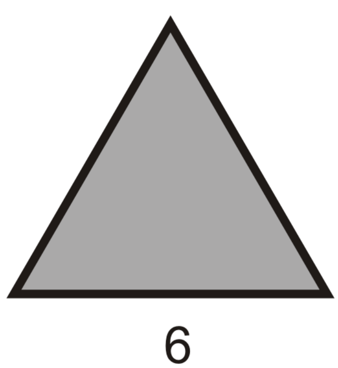
a. Both figures are squares.



b. Each vertex of the rhombus is 1.5 in from midpoints of the sides of the rectangle.



c. The figure is an equilateral triangle. (find the altitude)



d. Find the area of an equilateral triangle with side s.

Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Leave your answers in terms of π and reduced improper fractions.

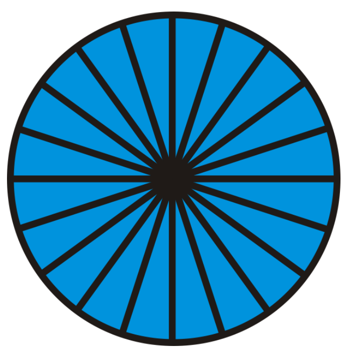
a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

b) A thin crust pizza has $\frac{1}{2}$ - in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

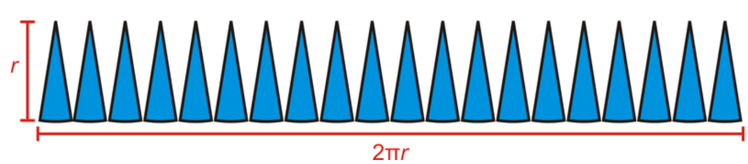
c) Which piece of pizza has more crust? A twelfth of the deep dish pizza or a fourth of the thin crust pizza?

Area of a Circle

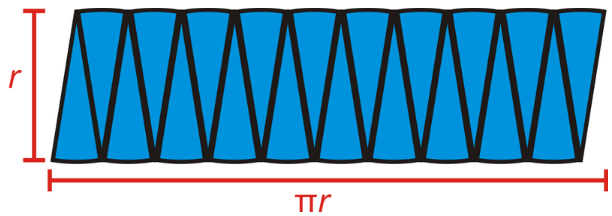
Recall in the previous section we derived π as the ratio between the circumference of a circle and its diameter. We are going to use the formula for circumference to derive the formula for area.



First, take a circle and divide it up into several wedges, or sectors. Then, unfold the wedges so they are all on one line, with the points at the top.



Notice that the height of the wedges is r, the radius, and the length is the circumference of the circle. Now, we need to take half of these wedges and flip them upside-down and place them in the other half so they all fit together.



Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html , by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If the diameter is 12 cm, then the radius is 6 cm. The area is $A = \pi(6^2) = 36\pi\ cm^2$.

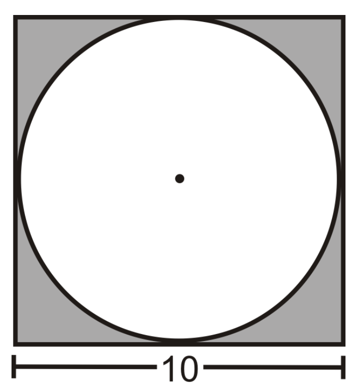
Example 2: If the area of a circle is 20π, what is the radius?

Solution: Work backwards on this problem. Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π, unless otherwise specified. In Example 2, the radius could be $\pm 2\sqrt{5}$, however the radius is always positive, so we do not need the negative answer.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is half the length of the side, or 5 cm.

$$A = \pi 5^2 = 25\pi\ cm$$

Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46\ cm^2$$

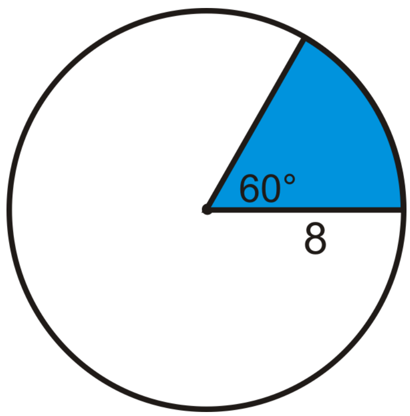
Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.

The area of a sector is a fractional part of the area of the circle, just like arc length is a fractional portion of the circumference.

Area of a Sector: If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π.



Solution: In the picture, the central angle that corresponds with the sector is 60°. 60° would be $\frac{1}{6}$ of 360°, so this sector is $\frac{1}{6}$ of the total area.

$$\textit{area of blue sector} = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$$

Another way to write the sector formula is $A = \frac{\textit{central angle}}{360^\circ} \cdot \pi r^2$.

Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x.

$$\begin{aligned}
8\pi &= \frac{x}{360^\circ} \cdot \pi 12^2 \\
8\pi &= \frac{x}{360^\circ} \cdot 144\pi \\
8 &= \frac{2x}{5^\circ} \\
x &= 8 \cdot \frac{5^\circ}{2} = 20^\circ
\end{aligned}$$

Example 7: The area of a sector of circle is 50π and its arc length is 5π. Find the radius of the circle.

Solution: First plug in what you know to both the sector formula and the arc length formula. In both equations we will call the central angle, "*CA*."

$$\begin{aligned} 50\pi &= \frac{CA}{360}\pi r^2 \\ 50 \cdot 360 &= CA \cdot r^2 \\ 18000 &= CA \cdot r^2 \end{aligned} \qquad\qquad \begin{aligned} 5\pi &= \frac{CA}{360} 2\pi r \\ 5 \cdot 180 &= CA \cdot r \\ 900 &= CA \cdot r \end{aligned}$$

Now, we can use substitution to solve for either the central angle or the radius. Because the problem is asking for the radius we should solve the second equation for the central angle and substitute that into the first equation for the central angle. Then, we can solve for the radius. Solving the second equation for CA, we have: $CA = \frac{900}{r}$. Plug this into the first equation.

$$\begin{aligned} 18000 &= \frac{900}{r} \cdot r^2 \\ 18000 &= 900r \\ r &= 20 \end{aligned}$$

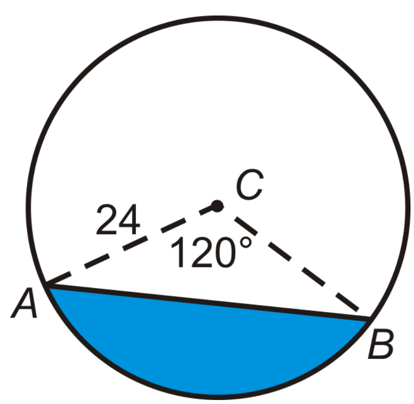
We could have also solved for the central angle in Example 7 once r was found. The central angle is $\frac{900}{20} = 45^\circ$.

Segments of a Circle

The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Example 8: Find the area of the blue segment below.



Solution: As you can see from the picture, the area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, we see that each half is a 30-60-90 triangle, where the radius is the hypotenuse. Therefore, the height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$A_{sector} = \frac{120}{360}\pi \cdot 24^2 = 192\pi \qquad A_{\triangle} = \frac{1}{2}\left(24\sqrt{3}\right)(12) = 144\sqrt{3}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

In the review questions, make sure you know how the answer is wanted. If the directions say "leave in terms of π and simplest radical form," your answer would be the first one above. If it says "give an approximation," your answer would be the second. It is helpful to leave your answer in simplest radical form and in terms of π because that is the most accurate answer. However, it is also nice to see what the approximation of the answer is, to see how many square units something is.

Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$. One-twelfth of the deep dish pizza has $\frac{15}{12}\pi$ or $\frac{5}{4}\pi$ in^2 of crust. One-fourth of the thin crust pizza has $\frac{31}{16}\pi$ in^2. To compare the two measurements, it might be easier to put them both into decimals. $\frac{5}{4}\pi \approx 3.93$ in^2 and $\frac{31}{16}\pi \approx 6.09$ in^2. From this, we see that one-fourth of the thin-crust pizza has more crust than one-twelfth of the deep dish pizza.

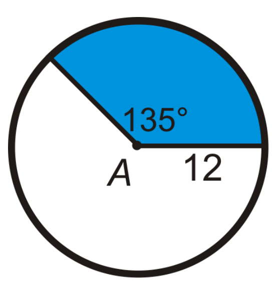
Review Questions

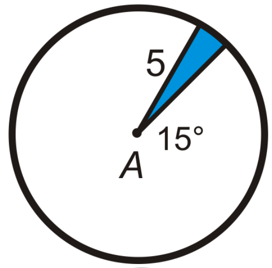
Fill in the following table. Leave all answers in terms of π.

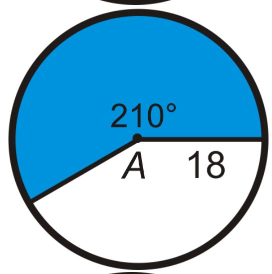
Table 10.2:

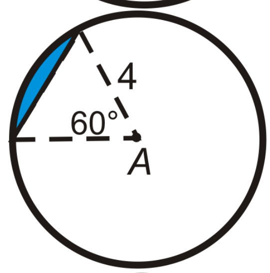
	radius	*Area*	*circumference*
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

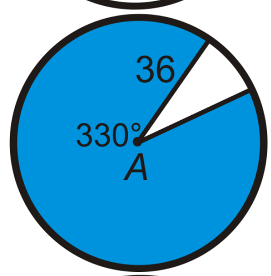
Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π. You may use decimals or fractions in your answers, but do not round.

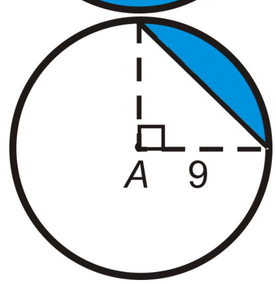
11. 

12. 

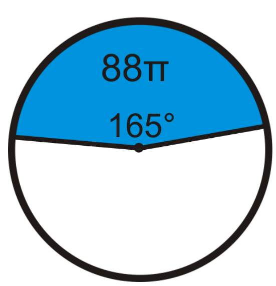
13. 

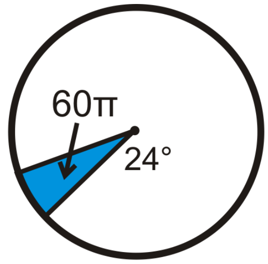
14. 

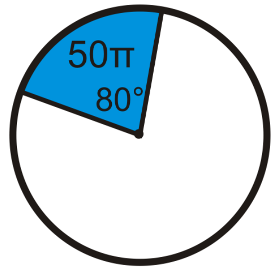
15. 

16. 

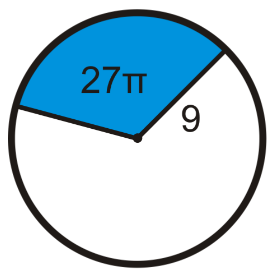
Find the radius of the circle. Leave your answer in simplest radical form.

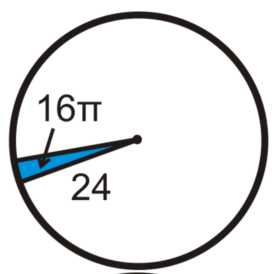
17. 

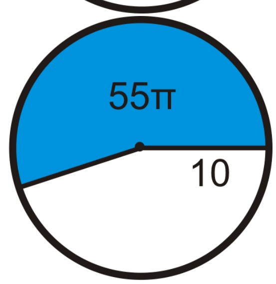
18. 

19. 

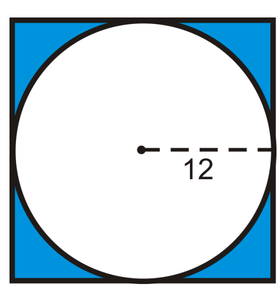
Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

20. 

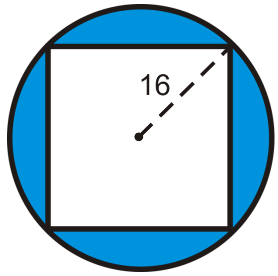
21. 

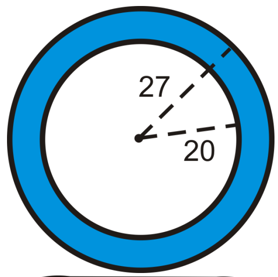
22. 

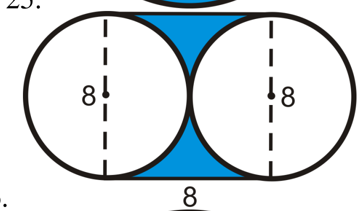
Find the area of the shaded region. Round your answer to the nearest hundredth.

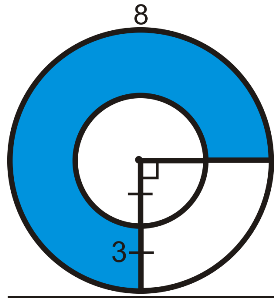
23. 

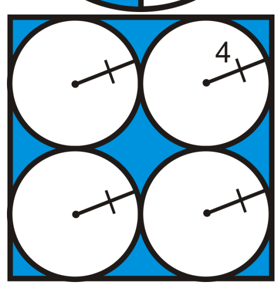
24. The quadrilateral is a square.



25. 

26. 

27. 

28. 

29. Carlos has 400 ft of fencing to completely enclose an area on his farm for an animal pen. He could make the area a square or a circle. If he uses the entire 400 ft of fencing, how much area is contained in the square and the circle? Which shape will yield the greatest area?
30. The area of a sector of a circle is 54π and its arc length is 6π. Find the radius of the circle.
31. The area of a sector of a circle is 2304π and its arc length is 32π. Find the central angle of the sector.

Review Queue Answers

a. $8^2 - 4^2 = 64 - 16 = 48$
b. $6(10) - \frac{1}{2}(7)(3) = 60 - 10.5 = 49.5$
c. $\frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$

d. $\frac{1}{2}(s)\left(\frac{1}{2}s\sqrt{3}\right) = \frac{1}{4}s^2\sqrt{3}$

10.6 Area and Perimeter of Regular Polygons

Learning Objectives

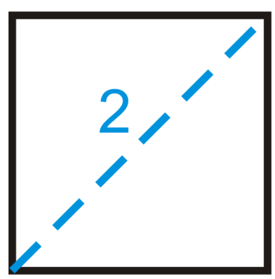
- Calculate the area and perimeter of a regular polygon.

Review Queue

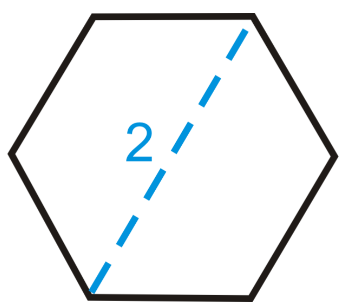
1. What is a regular polygon?

Find the area of the following regular polygons. For the hexagon and octagon, divide the figures into rectangles and/or triangles.

2.

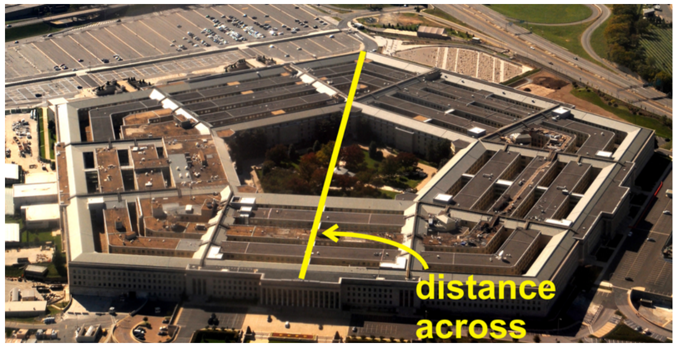


3.



4. Find the length of the sides in Problems 2 and 3.

Know What? The Pentagon in Arlington, VA houses the Department of Defense, is two regular pentagons with the same center. The entire area of the building is 29 acres (40,000 square feet in an acre), with an additional 5 acre courtyard in the center. The length of each outer wall is 921feet. What is the total distance across the pentagon? Round your answer to the nearest hundredth.



Perimeter of a Regular Polygon

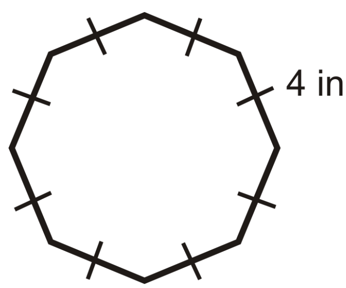
Recall that a regular polygon is a polygon with congruent sides and angles. In this section, we are only going to deal with regular polygons because they are the only polygons that have a consistent formula for area and perimeter. First, we will discuss the perimeter.

Recall that the perimeter of a square is 4 times the length of a side because each side is congruent. We can extend this concept to any regular polygon.

Perimeter of a Regular Polygon: If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

Example 1: What is the perimeter of a regular octagon with 4 inch sides?

Solution: If each side is 4 inches and there are 8 sides, that means the perimeter is 8(4 in) = 32 inches.

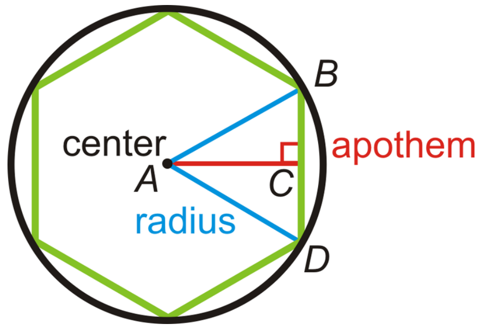


Example 2: The perimeter of a regular heptagon is 35 cm. What is the length of each side?

Solution: If $P = ns$, then 35 $cm = 7s$. Therefore, $s = 5$ cm.

Area of a Regular Polygon

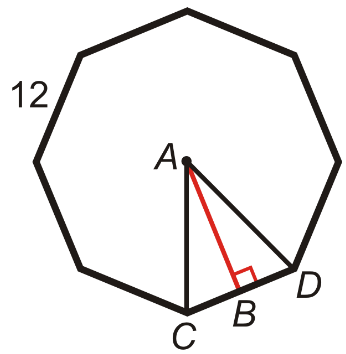
In order to find the area of a regular polygon, we need to define some new terminology. First, all regular polygons can be inscribed in a circle. So, ***regular polygons have a center and radius***, which are the center and radius of the circumscribed circle. Also like a circle, a regular polygon will have a central angle formed. In a regular polygon, however, ***the central angle is the angle formed by two radii drawn to consecutive vertices of the polygon.*** In the picture below, the central angle is $\angle BAD$. Also, notice that $\triangle BAD$ is an isosceles triangle. ***Every regular polygon with*** n ***sides is formed by*** n ***isosceles triangles.*** In a regular hexagon, the triangles are equilateral. The height of these isosceles triangles is called the ***apothem***.



Apothem: A line segment drawn from the center of a regular polygon to the midpoint of one of its sides.

We could have also said that the apothem is perpendicular to the side it is drawn to. By the Isosceles Triangle Theorem, the apothem is the perpendicular bisector of the side of the regular polygon. The apothem is also the height, or altitude of the isosceles triangles.

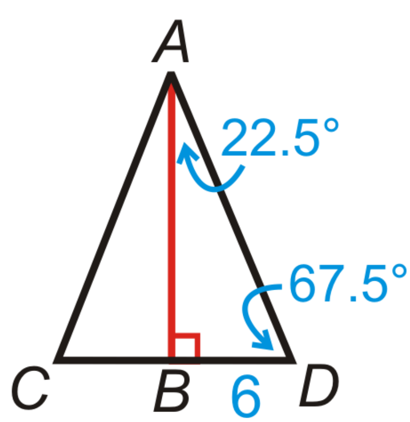
Example 3: Find the length of the apothem in the regular octagon. Round your answer to the nearest hundredth.



Solution: To find the length of the apothem, AB, you will need to use the trig ratios. First, find $m\angle CAD$. There are 360° around a point, so $m\angle CAD = \frac{360^\circ}{8} = 45^\circ$. Now, we can use this to find the other two angles in $\triangle CAD$. $m\angle ACB$ and $m\angle ADC$ are equal because $\triangle CAD$ is a right triangle.

$$\begin{aligned} m\angle CAD + m\angle ACB + m\angle ADC &= 180^\circ \\ 45^\circ + 2m\angle ACB &= 180^\circ \\ 2m\angle ACB &= 135^\circ \\ m\angle ACB &= 67.5^\circ \end{aligned}$$

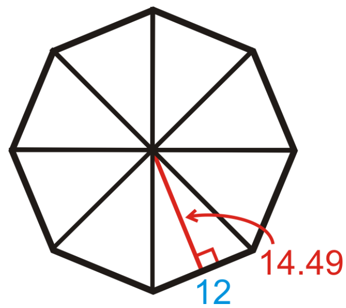
To find AB, we must use the tangent ratio. You can use either acute angle.



$$\tan 67.5^\circ = \frac{AB}{6}$$
$$AB = 6 \cdot \tan 67.5^\circ \approx 14.49$$

The apothem is used to find the area of a regular polygon. Let's continue with Example 3.

Example 4: Find the area of the regular octagon in Example 3.



Solution: The octagon can be split into 8 congruent triangles. So, if we find the area of one triangle and multiply it by 8, we will have the area of the entire octagon.

$$A_{octagon} = 8\left(\frac{1}{2} \cdot 12 \cdot 14.49\right) = 695.52\ units^2$$

From Examples 3 and 4, we can derive a formula for the area of a regular polygon.

The area of each triangle is: $A_\triangle = \frac{1}{2}bh = \frac{1}{2}sa$, where s is the length of a side and a is the apothem.

If there are n sides in the regular polygon, then it is made up of n congruent triangles.

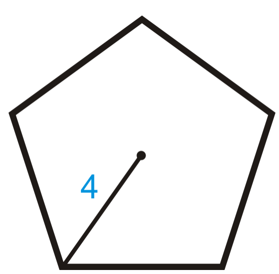
$$A = nA_\triangle = n\left(\frac{1}{2}sa\right) = \frac{1}{2}nsa$$

In this formula we can also substitute the perimeter formula, $P = ns$, for n and s.

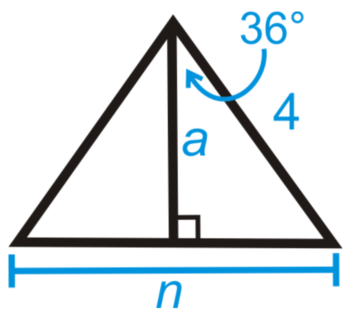
$$A = \frac{1}{2}nsa = \frac{1}{2}Pa$$

Area of a Regular Polygon: If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

Example 5: Find the area of the regular polygon with radius 4.



Solution: In this problem we need to find the apothem and the length of the side before we can find the area of the entire polygon. Each central angle for a regular pentagon is $\frac{360^\circ}{5} = 72^\circ$. So, half of that, to make a right triangle with the apothem, is 36°. We need to use sine and cosine.



$$\begin{aligned} \sin 36^\circ &= \frac{.5n}{4} \\ 4\sin 36^\circ &= \frac{1}{2}n \\ 8\sin 36^\circ &= n \\ n &\approx 4.7 \end{aligned} \qquad\qquad \begin{aligned} \cos 36^\circ &= \frac{a}{4} \\ 4\cos 36^\circ &= a \\ a &\approx 3.24 \end{aligned}$$

Using these two pieces of information, we can now find the area. $A = \frac{1}{2}(3.24)(5)(4.7) \approx 38.07\ units^2$.

Example 6: The area of a regular hexagon is $54\sqrt{3}$ and the perimeter is 36. Find the length of the sides and the apothem.

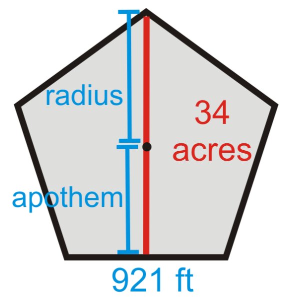
Solution: Plug in what you know into both the area and the perimeter formulas to solve for the length of a side and the apothem.

$$\begin{aligned} P &= sn \\ 36 &= 6s \\ s &= 6 \end{aligned} \qquad\qquad \begin{aligned} A &= \frac{1}{2}aP \\ 54\sqrt{3} &= \frac{1}{2}a(36) \\ 54\sqrt{3} &= 18a \\ 3\sqrt{3} &= a \end{aligned}$$

Know What? Revisited From the picture to the right, we can see that the total distance across the Pentagon is the length of the apothem plus the length of the radius. If the total area of the Pentagon is 34 acres, that is 2,720,000 square feet. Therefore, the area equation is $2720000 = \frac{1}{2}a(921)(5)$ and the apothem is 590.66 ft. To find the radius, we can either use the Pythagorean Theorem, with the apothem and half the length of a side or the sine ratio. Recall from Example 5, that each central angle in a pentagon is 72°, so we would use half of that for the right triangle.

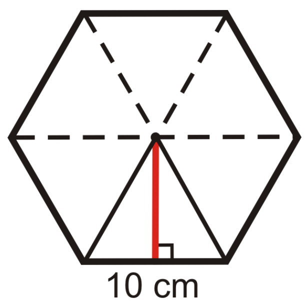
$$\sin 36^\circ = \frac{460.5}{r} \rightarrow r = \frac{460.5}{\sin 36^\circ} \approx 783.45\ ft.$$

Therefore, the total distance across is $590.66 + 783.45 = 1374.11\ ft$.



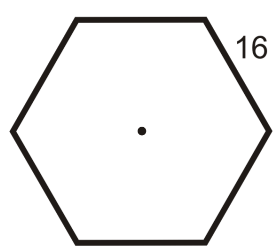
Review Questions

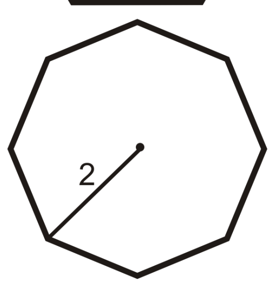
Use the regular hexagon below to answer the following questions. Each side is 10 cm long.

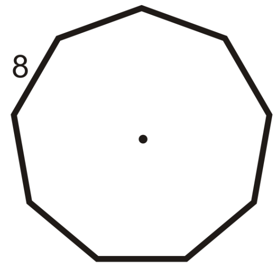


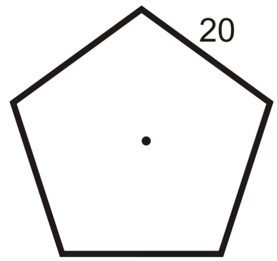
1. Each dashed line segment is $a(n)$ ________________.
2. The red line segment is $a(n)$ __________________.
3. There are _____ congruent triangles in a regular hexagon.
4. In a regular hexagon, all the triangles are _________________.
5. Find the radius of this hexagon.
6. Find the apothem.
7. Find the perimeter.
8. Find the area.

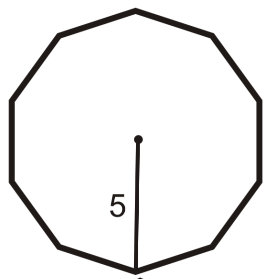
Find the area and perimeter of each of the following regular polygons. Round your answer to the nearest hundredth.

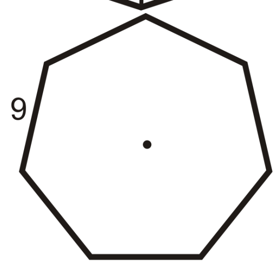
9. 

10. 

11. 

12. 

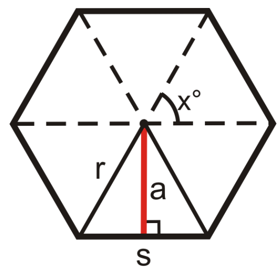
13. 

14. 

15. If the perimeter of a regular decagon is 65, what is the length of each side?
16. A regular polygon has a perimeter of 132 and the sides are 11 units long. How many sides does the polygon have?
17. The area of a regular pentagon is 440.44 in^2 and the perimeter is 80 in. Find the length of the pothem of the pentagon.
18. The area of a regular octagon is 695.3 cm^2 and the sides are 12 cm. What is the length of the apothem?

A regular 20-gon and a regular 40-gon are inscribed in a circle with a radius of 15 units.

19. Find the perimeter of both figures.
20. Find the circumference of the circle.
21. Which of the perimeters is closest to the circumference of the circle? Why do you think that is?
22. Find the area of both figures.
23. Find the area of the circle.
24. Which of the areas is closest to the area of the circle? Why do you think that is?
25. ***Challenge*** Derive a formula for the area of a regular hexagon with sides of length s. Your only variable will be s. HINT: Use 30-60-90 triangle ratios.
26. ***Challenge*** in the following steps you will derive an alternate formula for finding the area of a regular polygon with n sides.



We are going to start by thinking of a polygon with n sides as n congruent isosceles triangles. We will find the sum of the areas of these triangles using trigonometry. First, the area of a triangle is $\frac{1}{2}bh$. In the diagram to the right, this area formula would be $\frac{1}{2}sa$, where s is the length of a side and a is the length of the apothem. In the diagram, x represents the measure of the vertex angle of each isosceles triangle. a. The apothem, a, divides the triangle into two congruent right triangles. The top angle in each is $\frac{x^\circ}{2}$. Find $\sin\left(\frac{x^\circ}{2}\right)$ and $\cos\left(\frac{x^\circ}{2}\right)$. b. Solve your sin equation to find an expression for s in terms of r and x. c. Solve your cos equation to find an expression for a in terms of r and x. d. Substitute these expressions into the equation for the area of one of the triangles, $\frac{1}{2}sa$. e. Since there will be n triangles in an n-gon, you need to multiply your expression from part d by n to get the total area. f. How would you tell someone to find the value of x for a regular n-gon?

Use the formula you derived in problem 26 to find the area of the regular polygons described in problems 27-30. Round your answers to the nearest hundredth.

27. Decagon with radius 12 cm.
28. 20-gon with radius 5 in.
29. 15-gon with radius length 8 cm.
30. 45-gon with radius length 7 in.
31. What is the area of a regular polygon with 100 sides and radius of 9 in? What is the area of a circle with radius 9 in? How do these areas compare? Can you explain why?
32. How could you use the formula from problem 26 to find the area of a regular polygon given the number of sides and the length of a side? How can you find the radius?

Use your formula from problem 26 and the method you described to find r given the number of sides and the length of a side in problem 31 to find the area of the regular polygons below.

33. 30-gon with side length 15 cm.
34. Dodecagon with side length 20 in.

Review Queue Answers

a. A regular polygon is a polygon with congruent sides and angles.
b. $A = \left(\sqrt{2}\right)^2 = 2$
c. $A = 6\left(\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$
d. The sides of the square are $\sqrt{2}$ and the sides of the hexagon are 1 unit.

10.7 Chapter 10 Review

Keywords, Theorems and Formulas

Perimeter
The distance around a shape. Or, the sum of all the edges of a two-dimensional figure.

Area of a Rectangle
The area of a rectangle is the product of its base (width) and height (length) $A = bh$.

Perimeter of a Rectangle
$P = 2b + 2h$, where b is the base (or width) and h is the height (or length).

Perimeter of a Square
$P = 4s$

Area of a Square
$A = s^2$

Congruent Areas Postulate
If two figures are congruent, they have the same area.

Area Addition Postulate
If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Area of a Parallelogram
$A = bh$.

Area of a Triangle
$A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Area of a Trapezoid
The area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Rhombus
If the diagonals of a rhombus are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

Area of a Kite
If the diagonals of a kite are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

Area of Similar Polygons Theorem
If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

π
The ratio of the circumference of a circle to its diameter.

Circumference
If d is the diameter or r is the radius of a circle, then $C = \pi d$ or $C = 2\pi r$.

Arc Length
The length of an arc or a portion of a circle's circumference.

Arc Length Formula
length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

Area of a Circle
If r is the radius of a circle, then $A = \pi r^2$.

Sector of a Circle
The area bounded by two radii and the arc between the endpoints of the radii.

Area of a Sector
If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Segment of a Circle
The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Perimeter of a Regular Polygon
If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

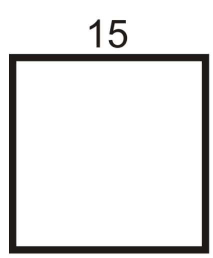
Apothem
A line segment drawn from the center of a regular polygon to the midpoint of one of its sides.

Area of a Regular Polygon
If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

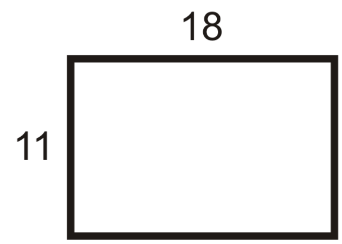
Review Questions

Find the area and perimeter of the following figures. Round your answers to the nearest hundredth.

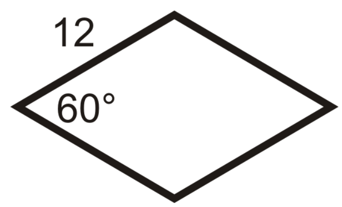
1. square



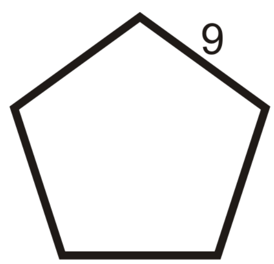
2. rectangle



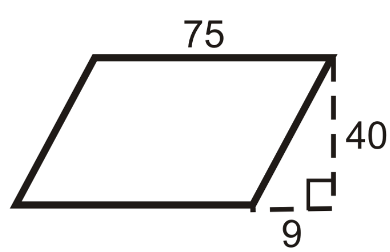
3. rhombus



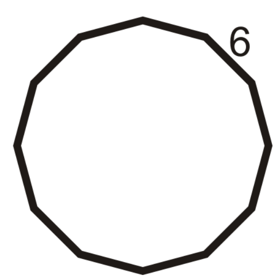
4. regular pentagon



5. parallelogram

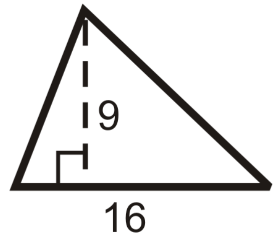


6. regular dodecagon

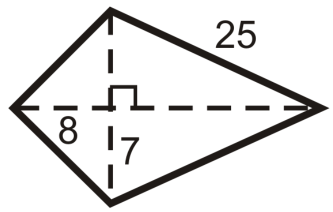


Find the area of the following figures. Leave your answers in simplest radical form.

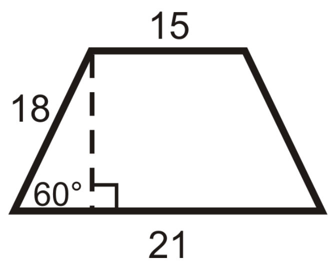
7. triangle



8. kite

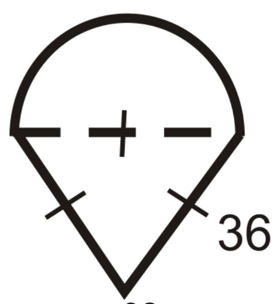


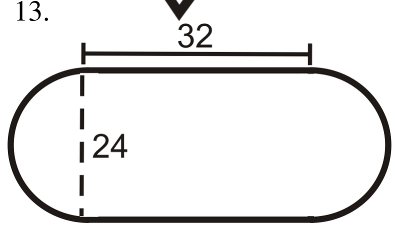
9. isosceles trapezoid



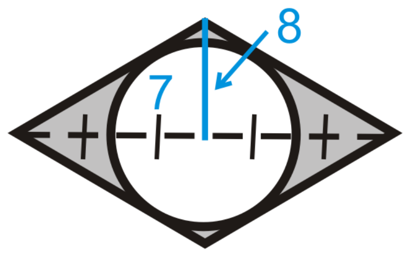
10. Find the area and circumference of a circle with radius 17.
11. Find the area and circumference of a circle with diameter 30.
12. Two similar rectangles have a scale factor $\frac{4}{3}$. If the area of the larger rectangle is 96 $units^2$, find the area of the smaller rectangle.

Find the area of the following figures. Round your answers to the nearest hundredth.

13. 

14. 

15. find the shaded area (figure is a rhombus)



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9695 .